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A New Theory of Collineations and their Lie Groups.

By H. B. Newson.

Introduction.

1. Among the groups of transformations discovered by Lie and used by him to illustrate his theory of continuous groups, the most important by far are the projective group and its subgroups. But a projective transformation, or collineation, may be studied synthetically as well as analytically, and hence arise the possibility and the desirability of a purely geometric theory of collineations and their groups. This consideration has led the writer to seek to develop such a theory. The investigation has been undertaken in the spirit of the following from Klein's "Erlangen Program":

> "Es ist zwar immer an der Forderung festzuhalten, dass man einen mathematischen Gegenstand noch nicht als erledigt betrachten soll, so lange er nicht begreiflich evident geworden ist, und es ist das Vordringen an der Hand des Formalismus eben nur ein erster aber schon sehr wichtiger Schritt."

Consequently, the present memoir is in no sense to be regarded as a contribution to Lie's theory of continuous groups, but its object is rather to set forth an independent geometric theory of the projective group of the plane. Starting with Lie's concepts of Transformation and Group, I have developed a new theory which is not based upon the infinitesimal transformation. A similar theory has already been published* by the writer for the projective transformations of the points on a line and the extension of the theory to the projective group in three dimensions is an easy and natural step.

Certain theorems and results reached in the above-mentioned paper will recur here, and reference will usually be made to that paper. Two important theorems on projective transformations in one dimension not found in the above paper will be frequently used and will be proved when introduced.

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* "Continuous Groups of Projective Transformations Treated Synthetically." Part I. Kansas University Quarterly, vol. IV, pp. 71–93.

§1.—PERSPECTIVE COLLINEATIONS; THEIR PROPERTIES AND LIE GROUPS.*

2. *Construction of a Perspective Collineation of the Plane.* Let p and p' be two planes intersecting in a line l and making any convenient angle with one another. From O, a point not in either plane, draw a bundle of rays each intersecting p and p'. Any ray a of the bundle cuts p and p' in the points P and P', a pair of corresponding points in the two planes. In this way the field of points on p is projected into a new field of points on p'. If, now, p' be revolved about l until it coincides with p, the two fields of points, i. e. the original field and its projection on p', may be considered as existing in the same plane.

The result of these two operations of projection and revolution is to shift the points of the plane p into new positions so that there is a one-to-one correspondence between the old and new positions of the points. Evidently, the right lines of p are also shifted into new positions and remain right lines after the operation. Any operation which transforms points into points and right lines into right lines is called a collineation, or projective transformation, of the plane. The most simple form of a plane collineation is the one constructed above, and is called a perspective collineation. We shall use S to denote a perspective collineation.

3. *Invariant Figure of S.* Any point or line of p which, after the operation S, coincides with its corresponding point or line, is said to be invariant under S. Evidently, from the construction, every point on l is an invariant point of S. The perpendicular from O on the plane bisecting the angle plp' cuts p in A and p' in A'; the revolution about l brings A' to A, so that A is also an invariant point of S.

Let B be any point on l; the lines g and g' joining A and B, A' and B respectively are corresponding lines in the two planes. The revolution about l makes g' coincide with g, so that g is an invariant line of S. In like manner every line through A is an invariant line of S. Every invariant line has on it two invariant points, viz. A and the point where it crosses l.

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*Many of the properties of perspective transformations contained in §1, were given in Professor A. Emch's dissertation, "On the Continuous Groups of Perspective Collineations in the Plane," published in the Kansas University Quarterly, Vol. V, pp. 1–36.

Every pencil of rays whose vertex is on l is invariant under S. Every invariant pencil of this sort has in it two invariant rays, viz. l and the ray through A. It should be noted that the invariant figure of S is a self-dualistic figure. The point A and the line l are called the vertex and axis respectively of the collineation S.

THEOREM 1.—*A perspective collineation S leaves invariant a self-dualistic figure consisting of a point A, every point on a line l, every line through A, and every pencil of rays with its vertex on l.*

4. *Characteristic Cross-Ratio of S.* Let q be the plane determined by O and l, and let r be the plane through l parallel to OA; let k be the cross-ratio of the pencil of planes $l(qrpp')$. Every line of the projecting bundle through O is cut by these four planes in ranges of four points, all of which have the same cross-ratio k. Any selected ray a of the bundle cuts these four planes in the points O, Q, P and P' respectively, and the cross-ratio $a(OQPP')$ is equal to k. If, now, every ray of the bundle be projected orthogonally on the plane p by lines parallel to OA, the four points O, Q, P, P' will be projected into A, B, P, P_1 and the cross-ratio $(ABPP_1) = k$. In like manner the cross-ratio of the two invariant points and every pair of corresponding points on every line through A is constant and equal to k.

Thus the perspective collineation S produces a one-dimensional projective transformation along every line of the plane through A, and the characteristic cross-ratio has the same value along all these lines. In like manner S produces a one-dimensional projective transformation in every pencil of lines with its vertex on l, and evidently the characteristic cross-ratio has the same value k in all these pencils.

THEOREM 2.—*A perspective collineation S is completely characterized by its invariant figure and a cross-ratio k which is the same along all invariant lines (except l) and through all invariant points (except A).*

5. *One-Parameter Group of Perspective Collineations.* The perspective collineation S is not the only one that can be constructed which leaves invariant the figure (A, l). Take O_1, a second point on OA, and proceed as before to construct a second collineation S_1. The invariant figures of S and S_1 will be the same, but the cross-ratios k and k_1 will not be the same; for the reason that the

cross-ratios of the axial pencils $l(qrpp')$ and $l(q_1rpp')$, where q_1 is the plane determined by l and O_1, are not the same. Thus we see that an unlimited number of perspective collineations can be constructed, leaving invariant the figure (A, l).

Consider, now, the operation whose effect on the points of the plane is equivalent to the effect of the successive operations of S and S_1. Since S and S_1 both transform points into points and lines into lines, their *resultant* must be a collineation; and, since S and S_1 both leave the figure (A, l) invariant, their resultant must also leave (A, l) invariant, and hence is a perspective collineation.

Let S transform the point P into P_1; the points P and P_1 are collinear with A and their joining line cuts l at some point B. Let S_1 transform the point P_1 to P_2; P_2 will also lie on the line AB. The resultant of S and S_1 is S_2, a perspective collineation which transforms the point P directly to P_2. This may be expressed by the equation $SS_1 = S_2$.

The characteristic cross-ratios of S, S_1 and S_2 are respectively $k = (ABPP_1)$, $k_1 = (ABP_1P_2)$ and $k_2 = (ABPP_2)$. Expanding these equations,

$$k = \frac{AP}{BP} : \frac{AP_1}{BP_1}, \quad k_1 = \frac{AP_1}{BP_1} : \frac{AP_2}{BP_2}, \quad k_2 = \frac{AP}{BP} : \frac{AP_2}{BP_2}.$$

Eliminating the fraction $\frac{AP_1}{BP_1}$ from the first and second by multiplication, we have

$$kk_1 = \frac{AP}{BP} : \frac{AP_2}{PP_2} = k_2.$$

Thus we see that the cross-ratio of S_2 is equal to the product of the cross-ratios of S and S_1.

We have now found that there are an unlimited number of perspective collineations leaving the figure (A, l) invariant, and that the resultant of any two of these is another of the same system. This is what is known as the *group property*, and hence we conclude that the aggregate of all perspective collineations leaving (A, l) invariant is a group. The natural parameter of this Lie group is the characteristic cross-ratio k.

THEOREM 3.—*The aggregate of all perspective collineations having the same vertex and axis is a Lie group of one parameter; the natural parameter of the group* *is the cross-ratio k; the cross-ratio of the resultant of any two collineations of the group is equal to the product of the cross-ratios of the components, i. e.* $k_2 = kk_1$.

6. *Properties of the Group $H_1(A, l)$.* For the sake of generality, we shall suppose that the point A and the line l are either real or imaginary, in general, imaginary. The points of the plane as P, P_1, etc., are also either real or imaginary, and hence the cross-ratio k is, in general, a complex number. Evidently there is a collineation in the group, which will be designated by $H_1(A, l)$, for every value of k.

The most important properties of the group may be enumerated as follows: If $k=1$ in the cross-ratio $k=(ABPP_1)$, P_1 must coincide with P, since A and B are separate points. Thus every point in the plane is an invariant point under this collineation, which is the identical collineation of the group.

Two collineations of the group for which k and k_1 have reciprocal values, have the effect of exactly neutralizing each other. Their resultant is the identical collineation, and they are called inverse collineations. Every collineation in the group has an inverse also in the same group.

There are two collineations in the group which are their own inverses, viz. those for which $k=1$ and $k=-1$; the first is the identical collineation and the second is called the involutoric collineation of the group. The effect of the involutoric collineation is to interchange every pair of corresponding points in the plane.

All collineations in the group for which $k = 1 + d$, where d is an infinitesimal complex number, are called infinitesimal collineations of the group. Since d may be written in the form $\delta e^{i\theta}$, where θ varies from 0 to 2π, we see that the group contains an infinite number of infinitesimal collineations. If the values of k are represented in the usual manner by points in the complex plane, the infinitesimal collineations are represented by points infinitely near to the unity point and are distinguished according to the directions of their points from the unit point.

Let the infinitesimal collineation $k = 1 + \delta e^{i\theta}$ be repeated n times; the resultant is a collineation of the group for which $k = (1 + \delta e^{i\theta})^n$, a number which can be made to differ as much as we please from 1 by taking n sufficiently large. If θ be constant, the locus of the point k, as n varies, is a logarithmic spiral about the origin. There is a separate spiral for each infinitesimal collineation of the group. Every collineation, whose cross-ratio is represented by a point on one of these spirals, is said to be generated from the infinitesimal collineation of the spiral. These spirals all pass through the unit point and lie infinitely close together. There are two families of these spirals, one right-handed and the other left-handed. They cover twice over the entire plane, and hence every collineation in the group, except those for which $k=0$ and $k=\infty$, may be generated from at least two infinitesimal collineations; those for which k is real and positive may be generated from three infinitesimal collineations.*

The two collineations for which $k=0$ and $k=\infty$ are essentially different in character from all other collineations in the group. Their corresponding points in the plane of k are the points toward which all the spirals converge without ever reaching them. The effect of the collineation $k=0$ is to transform every point of the plane, except A, to a point on l; the effect of the other one, $k=\infty$, is to transform every point of the plane, except those on l, to A. These two *pseudo*-transformations are called the essentially singular collineations of the group.

A Lie group is defined to be continuous when every transformation in it can be generated from an infinitesimal transformation of the group. Hence the group of perspective collineations $H_1(A, l)$ is continuous everywhere except for $k=0$ and $k=\infty$.†

THEOREM 4.—*The one-parameter group of perspective collineations $H_1(A, l)$ is continuous everywhere in the plane of its complex parameter k, except at $k=0$ and $k=\infty$.*

7. *Resultant of Two Coaxial Perspective Collineations.* Let S and S_1 be two perspective collineations having the same axis l and their vertices at A and A_1 respectively. Let the line joining A and A_1 be l', cutting the axis l at B; let k and k_1 be the cross-ratios of S and S_1 respectively. The resultant of S and S_1 is a collineation and leaves invariant every point on the axis l. The line l' is an invariant line of both S and S_1, and is, therefore, also an invariant line of S_2.

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* For a fuller discussion of this theory, see the Bulletin of the American Mathematical Society (2), vol. IV, pp. 113–114.

† In his definition of a projective transformation, Lie explicitly lays down the condition that the determinant of the transformation shall not vanish; this condition excludes from consideration the transformations of the group for which $k=0$ and ∞. Lie's group thus defined is everywhere continuous for the reason that he has excluded the discontinuity points.

Along the line l' we have two one-dimensional projective transformations whose invariant points are (A, B) and (A_1, B) respectively. Their resultant is a one-dimensional projective transformation whose invariant points are A_2 and B, where A_2 is some other point on l'. Since S_2 leaves invariant every point on l and the point A_2, it must be a perspective collineation, and also must leave invariant every line through A_2 and every pencil with vertex on l.

Since S_2 is a perspective collineation, it is characterized by a constant cross-ratio k_2, which we wish to determine. Consider the pencil of lines through the point B; the invariant rays of this pencil in all three collineations S, S_1 and S_2 are l and l'. The one-dimensional transformations of this pencil due to S and S_1 have cross-ratios k and k_1; hence, the cross-ratio of their resultant is $k_2 = kk_1$. Hence, the cross-ratio of one of the invariant pencils of S_2 is $k_2 = kk_1$; but it has been proved that a perspective collineation is characterized by a constant cross-ratio in all invariant pencils and along all invariant lines. Therefore, the characteristic cross-ratio of S_2 is everywhere equal to the product of the cross-ratios of S and S_1.

THEOREM 5.—*The resultant of two coaxial perspective collineations is a perspective collineation having the same axis; the characteristic cross-ratio of the resultant is equal to the product of the cross-ratios of the components.*

8. *Resultant of Two One-Dimensional Transformations with a Common Invariant Point.* From the last result we are able to infer an important theorem for one-dimensional projective transformations. Along the line l' the two one-dimensional transformations have one invariant point B in common, and it was shown above that the cross-ratio of the resultant is equal to the product of the cross-ratios of the components. In every pencil with vertex on l, except at B, the two one-dimensional projective transformations have only one invariant ray l in common and the cross-ratio of the resultant is equal to the product of the cross-ratios of the components. We know also that this law of combination of the cross-ratios always holds good when the two invariant elements are in common. We are able, therefore, to state the following theorem :[*]

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[*] This theorem properly belongs to the subject of one-dimensional projective transformations, but I had not observed it at the time the former paper was written. An analytic proof was given by the writer in Bulletin American Mathematical Society (2), vol. IV, p. 110. It was there shown that the

THEOREM 6.—*The cross-ratio of the resultant of two one-dimensional projective transformations, which have one or both invariant elements in common, is equal to the product of the cross-ratios of the components.*

9. *Invariant Figure and Properties of an Elation.* The vertex A_2 of the resultant of two coaxial perspective collineations may be anywhere on the line l'. We wish to examine the case when A_2 coincides with B. The cross-ratio of the resultant is given by $(A_2BPP_2) = kk_1$. When $kk_1 = k_2 = 1$, either $P = P_2$ or $A_2 = B$. In the first case, the resultant is the identical collineation; in the second case, it is a special kind of perspective collineation in which the vertex A falls on the axis l. This special kind of perspective collineation is called an Elation. It is the resultant of two coaxial perspective collineations having different vertices and reciprocal values of k.

In the case of an elation, there is no longer a characteristic cross-ratio, for there are only three elements, e. g. B, P and P_1; but a new relation can be found between the segments BP and BP_1. We shall suppose, for simplicity, that the line l' is perpendicular to l. When A and B are distinct points, we have $k = (ABPP_1)$, whence $(APBP_1) = 1 - k$. Expanding, we get

$$\frac{AB}{PB}\cdot\frac{PP_1}{AP_1} = 1 - k \quad \text{or} \quad \frac{PP_1}{PB.AP_1} = \frac{1-k}{AB}.$$

Writing $AP_1 - AP$ for PP_1, we get

$$\frac{AP_1 - AP}{PB.AP_1} = \frac{1-k}{AB}.$$

Taking the limit of both sides of the last equation as A approaches B and setting $\lim\limits_{A=B}\dfrac{1-k}{AB} = \alpha$, we find the relation

$$-\frac{1}{BP} + \frac{1}{BP_1} = \alpha \quad \text{or} \quad \frac{1}{BP_1} = \frac{1}{BP} + \alpha,$$

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cross-ratio k_2 of the resultant of any two one-dimensional transformations is given by

$$\frac{k_2+1}{\sqrt{k_2}} = \frac{kk_1 - k - k_1 + 1}{\sqrt{kk_1}}\lambda + \frac{k+k_1}{\sqrt{kk_1}},$$

where $\lambda = (A_1ABB_1)$, one of the cross-ratios of the two pairs of invariant points (AB) and (A_1B_1). When $A = A_1$ or $B = B_1$, or both are simultaneously equal, $\lambda = 1$ and $k_2 = kk_1$.

where α is a constant for all pairs of corresponding points, because k is such a constant. Let P_i be the point on l' into which the point at infinity is transformed. Then

$$\frac{1}{BP_i} = \frac{1}{\infty} + \alpha;$$

thus α is the reciprocal of the segment BP_i.

The elation S' leaves invariant every point on the axis l; and every invariant pencil with vertex on l has only one invariant ray, viz. l. Let us consider one of these pencils with vertex at C on l. We have in the general case $C(ABPP_1) = k$, whence $C(APBP_1) = 1 - k$. Expanding this,

$$\frac{\sin(ACB)}{\sin(PCB)} \cdot \frac{\sin(PCP_1)}{\sin(ACP_1)} = 1 - k \quad \text{or} \quad \frac{\sin(PCP_1)}{\sin(PCB) . \sin(ACP_1)} = \frac{1-k}{\sin(ACB)}.$$

For PCP_1 write $BCP_1 - BCP$, whence,

$$\frac{\sin(BCP_1)\cos(BCP) - \cos(BCP_1)\sin(BCP)}{\sin(PCB) . \sin(ACP_1)} = \frac{1-k}{\sin(ACB)}.$$

Take the limit of both sides as A approaches B and put

$$\lim_{A=B} \frac{1-k}{\sin(ACB)} = \lim_{A=B} \frac{AC(1-k)}{AB} = \alpha . BC.$$

We thus find $-\cot(BCP) + \cot(BCP_1) = BC\alpha$; let $BCP = \theta$, $BCP_1 = \theta_1$, and $BC = a$; we then have

$$\cot\theta_1 = \cot\theta + a\alpha.$$

This relation is constant for all pairs of corresponding rays in any given pencil whose vertex is at a distance a from B.

When the vertex of the pencil is at B, $a = 0$; in this case $\theta_1 = \theta$, and every ray of this pencil is an invariant ray of S'. This is as it should be, for in a perspective collineation S, of which S' is a special case, every ray through the vertex is invariant.

Along every ray of the pencil through B we have a one-dimensional projective transformation of the second* kind t', viz. with only one invariant point B.

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* There are two kinds of one-dimensional projective transformations; the first kind has two invariant elements, and is designated by t, the second kind has one invariant element, and is designated by t'.

Along l'', any ray of this pencil making an angle ϕ with l, the characteristic constant is not the same as along l'. Let g be the line through P parallel to l; S' transforms g into g_1, the line through P_1 parallel to l. Let g and g_1 cut l' in Q and Q_1; then Q is transformed by S' into Q_1. Since $AP = AQ \sin\phi$ and $AP_1 = AQ_1 \sin\phi$, we have the relation

$$\frac{1}{AQ_1} = \frac{1}{AQ} + \alpha \sin\phi .$$

An elation S' is thus seen to have properties quite different from those of a perspective collineation S. The invariant figure of S' consists of all points on l and all lines through A, some point of l. S' produces along all lines through A, and in all pencils with vertices on l, one-dimensional projective transformations of the second kind, viz. with only one invariant element. It is fully characterized by its invariant figure and a constant α.

THEOREM 7.—*The resultant of two coaxial perspective collineations with different vertices and reciprocal cross-ratios is an elation, having the same axis and its vertex collinear with the vertices of the components. An elation is completely determined by its invariant figure and a characteristic constant* α.

10. *The Group* $H'_1(Al)$ *and its Properties.* The aggregate of all elations with the same invariant figure Al has the group property and forms a group as we proceed to show. Let S' and S'_1 be two such elations whose constants are α and α_1 respectively. The effect of S' on a point P of the plane is to move it to P_1 such that $\frac{1}{AP_1} = \frac{1}{AP} + \alpha \sin\theta$, where θ is the angle made by the line AP and l. The effect of S'_1 on P_1 is to move it to P_2 such that $\frac{1}{AP_2} = \frac{1}{AP_1} + \alpha_2 \sin\theta$. Eliminating AP_1 from these equations, we find $\frac{1}{AP_2} = \frac{1}{AP} + (\alpha + \alpha_1)\sin\theta$; hence, the resultant of S' and S'_1 is an elation with the same invariant figure and a characteristic constant $\alpha_2 = \alpha + \alpha_1$. Therefore, the aggregate of all elations with invariant figure Al form a group $H'_1(Al)$.

The group $H'_1(Al)$ contains an elation corresponding to every value of α, which is, in general, a complex number. Let the values of α be represented by points in the complex plane. The following properties of the group may be enunciated:

The group $H_1'(Al)$ contains one identical elation for which $\alpha = 0$. The elations of the group for which α and α_1 have equal numerical values but of opposite signs, are inverse to one another, and their resultant is the identical elation of the group. Every elation in the group has its inverse in the same group. The group contains ∞^1 infinitesimal elations represented by the points adjacent to the origin; the corresponding values of α are $\delta e^{i\theta}$. Let one of these infinitesimal elations be repeated n times; its final resultant is an elation of the group for which $\alpha = n\delta e^{i\theta}$. If θ is a constant, the locus of the point α as n varies is a half ray proceeding from the origin and making an angle θ with the axis of real numbers. Every elation of the group which is represented by a point on this half ray is generated from this infinitesimal elation. Each infinitesimal elation of the group has its corresponding half ray, and these rays lie infinitesimally close together and cover the entire plane. Thus every elation in the group (except that for which $\alpha = \infty$) is generated from an infinitesimal elation of the group. Hence the group is continuous everywhere in the plane of its parameter α except at the infinity point. The elation corresponing to $\alpha = \infty$ is an essentially singular elation of the group; its effect is to transform every point in the plane, except those on the line l, to the vertex A.

THEOREM 8.—*The one-parameter group of elations is continuous for all values of the parameter α except for $\alpha = \infty$.*

11. *Number of Perspective Collineations and Elations.* The number of perspective collineations in the plane is determined by the number of one-parameter groups of the kind $H_1(A, l)$ and $H_1'(Al)$. There are ∞^2 lines and ∞^2 points in the plane, and hence there are ∞^4 combinations of line and point such as (A, l) and ∞^3 lineal elements (Al) in the plane. Each of these figures is the invariant figure of a one-parameter group of perspective collineations; in the case that the point A falls on l, the resulting group is a group of elations. Hence, there are in the plane ∞^4 one-parameter groups of perspective collineations, ∞^3 of which are groups of elations. Since each one-parameter group contains ∞^1 collineations, there are ∞^5 perspective collineations in the plane, ∞^4 of which are elations.

We shall next concern ourselves with the question whether these ∞^5 collineations form a five-parameter group, and what groups of order higher than the first may be built up out of these ∞^4 one-parameter groups. We shall first look for other groups of elations.

12. *The Group $H'_2(l)$.* Let us consider the aggregate of all coaxial one-parameter groups of elations, l being the axis. Every point on l is the vertex of such a group, and hence there are ∞^2 elations leaving invariant every point on l. To ascertain if this system of elations has the group property, we find the resultant of any two elations S' and S'_1 of the system. Let S' be taken from the group whose vertex is at A and S'_1 from the group with vertex at A_1; let their characteristic constants be α and α_1 respectively.

Let us first examine the effect of S' and S'_1 on the pencil of lines through C, a chosen point on l. S' produces in the pencil (C) a one-dimensional projective transformation of the second kind of which l is the single invariant ray. It p and p_1 are two corresponding rays of the pencil making angles θ and θ_1 with l, we have

$$\cot\theta_1 = \cot\theta + a\alpha.$$

Let S'_1 transform p_1 to p_2, where the angle between p_2 and l is θ_2; then we have the relation

$$\cot\theta_2 = \cot\theta_1 + a_1\alpha_1.$$

Eliminating θ_1 from these equations by addition, we have

$$\cot\theta_2 = \cot\theta + a_2\alpha_2,$$

where $a_2\alpha_2 = a\alpha + a_1\alpha_1$. This shows that S'_2, the resultant of S' and S'_1, is an elation whose axis is l and whose vertex is some point on l; for it produces a one-dimensional projective transformation of the second kind in every pencil with vertex on l. Thus the group property is established and we have found a two-parameter group of elations, the parameters being a and α; this group will be designated by $H'_2(l)$.

Let us next consider the effect of S' and S'_1 on the pencil whose vertex is at infinity on l. Since the rays of the pencil are parallel to l, S' transforms g into g_1 such that $1/p_1 = 1/p + \alpha$, where p and p_1 represent the perpendiculars from l on the lines g and g_1. S'_1 transforms g_1 into g_2 so that $1/p_2 = 1/p_1 + \alpha_1$; eliminating p_1, we get $1/p_2 = 1/p + \alpha_2$, where $\alpha_2 = \alpha + \alpha_1$. Thus the values of a_2 and α_2 are obtainable from the equations

$$a_2\alpha_2 = a\alpha + a_1\alpha_1 \quad \text{and} \quad \alpha_2 = \alpha + \alpha_1.$$

Since the values of a_2 and α_2 are unaltered when a and a_1, α and α_1 are interchanged, it follows that the elations of the group $H_2'(l)$ are commutative.

THEOREM 9.—*The doubly infinite system of elations having a common axis l forms a two-parameter group $H_2'(l)$ in which the elations are commutative.*

13. *The Group $H_2'(A)$.* We next consider the aggregate of all one-parameter groups of elations which have a common vertex A and whose axes are in turn every line through A. We shall find that this system also possesses the group property.

Since every elation of the system leaves invariant the point A and every line through A, it follows that the resultant of any two elations of the system must also leave this point and pencil invariant. Take any two elations of the system S' and S_1' whose axes are the lines l and l_1 and whose constants are α and α_1 respectively. Along every invariant ray of the pencil the resultant of the two one-dimensional projective transformations of the second kind with common invariant point at A, is also one of the second kind having the same invariant point A. Hence S_2', the resultant of S' and S_1', is also an elation with vertex at A and whose axis is some line l_2 through A. The group property is thus established.

In order to determine the value of α_2 and the position of the axis l_2 of S_2', we proceed as follows: Along any chosen ray g of the pencil through A, the constants of S', S_1' and S_2' are $\alpha \sin\phi$, $\alpha_1 \sin\phi_1$ and $\alpha_2 \sin\phi_2$, where ϕ, ϕ_1 and ϕ_2 are the angles which g makes with l, l_1 and l_2 respectively. But the constant of the resultant is equal to the sum of the constants of the components; hence, $\alpha_2 \sin\phi_2 = \alpha \sin\phi + \alpha_1 \sin\phi_1$.

Along l_2 the resultant is an identical transformation and its constant is 0; hence, we have $\alpha \sin(\phi_2 - \phi) + \alpha_1 \sin(\phi_2 - \phi_1) = 0$. Expanding and collecting, we get

$$(\alpha \cos\phi + \alpha_1 \cos\phi_1) \sin\phi_2 = (\alpha \sin\phi + \alpha_1 \sin\phi_1) \cos\phi_2 = \alpha_2 \sin\phi_2 \cos\phi_2, \qquad \text{(a)}$$

whence, $\alpha \cos\phi + \alpha_1 \cos\phi_1 = \alpha_2 \cos\phi_2$.

From the two equations

$$\alpha_2 \sin\phi_2 = \alpha \sin\phi + \alpha_1 \sin\phi_1, \quad \text{and} \quad \alpha_2 \cos\phi_2 = \alpha \cos\phi + \alpha_1 \cos\phi_1, \qquad \text{(b)}$$

we can calculate the values of α_2 and ϕ_2. Thus:

$$\alpha_2 = \sqrt{\alpha^2 + \alpha_1^2 + 2\alpha\alpha_1 \cos(\phi - \phi_1)}. \tag{c}$$

When $\phi_1 = \phi$, this reduces to $\alpha_2 = \alpha + \alpha_1$, which is as it should be.

Evidently, from equations (b), the elations of the group $H_2'(A)$ are commutative.

THEOREM 10.—*The doubly infinite system of elations having a common vertex forms a two-parameter group $H_2'(A)$ in which the elations are commutative.*

14. *No other Groups of Elations.* There are only two other relative positions besides those discussed above in which two lineal elements Al and A_1l_1 can be placed:

1st. Let S' and S_1' be two elations whose lineal elements Al and A_1l_1 are so situated that A_1 lies on l. The effect of S' is to leave invariant A_1 and l, but no other line through A_1. If, now, S_1' operates on the points of the plane, it leaves invariant A_1 and l, but no other point on l. Hence, the resultant of S' and S_1' leaves invariant A_1 and l but no other points on l and no other lines through A_1. Therefore, the resultant S_2', although a collineation, is not an elation; and the aggregate of all elations whose invariant figures have this relative position, does not form a group. The resulting collineation in this case is very important, and will be discussed later. See article 62.

2d. In the second case, let S' and S_1' be two elations whose invariant lineal elements Al and A_1l_1 are in the most general relative position. Let l and l_1 intersect in A_2, and let the join of A and A_1 be l_2. The resultant of S' and S_1' leaves invariant A_2 and l_2, but not every point on l_2. Hence, the resultant S, though a collineation, is neither an elation nor a perspective collineation. Therefore, the aggregate of all the ∞^4 elations of the plane does not form a group.

Thus we see that there are only three kinds of groups of elations, viz. $H_1'(Al)$, $H_2'(l)$ and $H_2'(A)$.

15. *The Groups $H_3(l)$ and $H_2(lBl')$.* We found in article 7 that the resultant of S and S_1, two coaxial perspective collineations, is, in general, a perspective collineation S_2 whose axis is also l and whose vertex A_2 is collinear with A and A_1; and, furthermore, that $k = kk_1$. From these results we may draw important conclusions: 1st, the aggregate of all coaxial perspective collineations has the group property and forms a Lie group of three parameters, the param-

eters being k and the two coordinates of the vertex A; 2d, the aggregate of all coaxial perspective collineations whose vertices are collinear, forms a two-parameter Lie group, the parameters being k and one coordinate of A.

The first group, designated by $H_3(l)$, contains ∞^2 one-parameter subgroups $H_1(A, l)$, one for each point in the plane. It also contains ∞^2 subgroups of the second variety, designated by $H_2(lBl')$, one for each line l' in the plane. But two of these latter groups always contain one one-parameter subgroup in common, viz. $H_1(A_2, l)$, where A_2 is the intersection of l' and l'_1.

The group $H_2(lBl')$ contains ∞^1 subgroups $H_1(A, l)$, one for each point on l'; it also contains one subgroup of elations, viz. $H'_1(Bl)$; the group $H_3(l)$ contains one two-parameter subgroup of elations, viz. $H'_2(l)$.

Within the groups $H_3(l)$ and $H_2(lBl')$ the collineations are, in general, not commutative; the value of k_2 is independent of the order of S and S_1, but the position of A_2 is not independent of the order of S and S_1.

THEOREM 11.—*The aggregate of all coaxial perspective collineations forms a three-parameter group $H_3(l)$; the aggregate of all coaxial perspective collineations whose vertices are collinear forms a two-parameter group $H_2(lBl')$. The collineations of these groups are not commutative.*

16. *The Groups $H_3(A)$ and $H_2(Al'B)$.* We next proceed to find the resultant of two perspective collineations S and S_1 which have a common vertex A and whose axes l and l_1 intersect in B. It is evident that the resultant S_2 leaves invariant A, B, and every line through A. Also the two one-dimensional projective transformations in the pencil through B have one invariant l' in common, and hence their resultant leaves invariant l' and some second ray l_2 through B. Since A, l' and every line through A are invariant, every point on l_2 must also be invariant under S_2; and hence S_2 is also a perspective collineation with vertex at A. Thus the group property is established, and we infer that all perspective collineations with vertices at A form a three-parameter group $H_3(A)$. Along AB and also through B we have $k_2 = kk_1$, and hence the cross-ratio of S_2 is everywhere given by $k_2 = kk_1$.

From the above discussion we also infer that all perspective collineations whose vertices coincide and whose axes are concurrent in B form a two-parameter group, $H_2(Al'B)$. Evidently the group $H_2(Al'B)$ contains one subgroup of elations, $H'_1(Al')$; and $H_3(A)$ contains $H'_2(A)$ as a subgroup. The collineations of these groups are not commutative.

THEOREM 12.—*The aggregate of all perspective collineations which have a common vertex forms a three-parameter group, $H_3(A)$; within the group $H_3(A)$ all collineations whose axes are concurrent form a two-parameter subgroup, $H_2(Al'B)$. The collineations in these groups are not commutative.*

17. *No other Groups of Perspective Collineations.* We now proceed to show that the resultant of two perspective collineations, whose invariant figures are in any relative positions different from those discussed above, is not a perspective collineation, and hence there are no other varieties of groups except the eight already found. There are four other cases to be considered where the relative position of the invariant figures of S and S_1 are different from those above.

1st. Let the invariant figures (A, l) and (A_1, l_1) be so situated that A falls on l_1 and A_1 on l; let the intersection of l and l_1 be B_2 and the join of A and A_1 be l_2. The resultant of S and S_1 leaves invariant the triangle (AA_1B_2), but not every point on any side of it; hence, the resultant is not a perspective collineation, and all perspective collineations in this relative position do not form a group.

2d. Let the point A fall on l_1 but A_1 not on l, or vice versa; along l_2, the join of A and A_1, the two one-dimensional projective transformations result in another of the same kind, leaving invariant A and C, some other point on l_2. Thus the resultant leaves invariant a triangle (ACB_2) and is, in general, not a perspective collineation.

3d. Let the points A and A_1 of the first case be collinear with B_2; the resultant collineation leaves invariant B_2, A_2 some point on AA_1 and some line l_2 through B_2. But every line through A_2 will not be invariant, nor every point on l_2. Hence, the resultant is, in general, not a perspective collineation.

4th. Let the figures (A, l) and (A_1, l_1) be in the most general relative position so that l and l_1 meet in B_2, the join of A and A_1 does not pass through B_2 and A and A_1 do not fall on l_1 and l respectively. The resultant collineation leaves invariant a triangle $(A_2B_2C_2)$, where A_2 and C_2 are points on l_2, but it is not a perspective collineation. Hence, the aggregate of all ∞^5 perspective collineations of the plane do not form a five-parameter group.

Therefore, there are no four- or five-parameter groups of perspective collineations, and no other kinds of one-, two- and three-parameter groups besides those discussed above.

18. *Dualistic and Self-Dualistic Groups.* From the dualistic character of a collineation, we infer that its fundamental invariant figure is self-dualistic. When there exists a group of collineations, leaving a certain figure invariant, there must also exist a second group leaving invariant a figure dualistic to the first. Two such groups are said to be dualistic to one another; if the invariant figure of the group is a self-dualistic figure, the group is called a self-dualistic group.

As examples, we may cite the following pairs of dualistic groups: $H_3(A)$ and $H_3(l)$, $H_2(lBl')$ and $H_2(Al'B)$, $H'_2(A)$ and $H'_2(l)$; the one-parameter groups $H_1(A, l)$ and $H'_1(Al)$ are self-dualistic groups.

This principle holds for all collineations of whatever kind, and hereafter, when the existence of a certain group has been proved, it will be assumed, without further proof, that the dualistic group also exists and has properties dualistic to the first.

19. *Resumé of Perspective Collineations.* In the following list, the structure of all groups of perspective collineations is indicated. A dash above a letter indicates that the line or point thus marked takes on different positions in the invariant figure of the group. Thus $H'_2(l) = \infty\, H'_1(\bar{A}l)$ indicates that the point A takes all positions on the line l. Dualistic groups are bracketed together, and self-dualistic groups are bracketed alone.

$$\{H'_1(Al)\},$$
$$\{H_1(A, l)\},$$
$$\left\{\begin{array}{ll} H'_2(l) & = \infty^1 H'_1(\bar{A}l) \\ H'_2(A) & = \infty^1 H'_1(A\bar{l}) \end{array}\right.$$
$$\left\{\begin{array}{ll} H_2(lBl') & = \infty^1 H_1(\bar{A}, l) + H'_1(Bl'), \\ H_2(ABl') & = \infty^1 H_1(A, \bar{l}) + H'_1(Al'), \end{array}\right.$$
$$\left\{\begin{array}{ll} H_3(l) & = \infty^2 H_1(\bar{A}, l) + H'_2(l), \\ H_3(A) & = \infty^2 H_1(A, \bar{l}) + H'_2(A). \end{array}\right.$$

§2.—COLLINEATIONS WITH INVARIANT TRIANGLE. TYPE I.

A.—*The Group $G_2(ABC)$ and its Subgroups.*

20. *The Two-Parameter Group $G_2(ABC)$.* The most general form of a collineation in the plane leaves invariant a triangle. Such a collineation is the

resultant of two perspective collineations whose invariant figures are so situated that the vertex of each is on the axis of the other. Thus, let S and S_1 be two such perspective collineations whose invariant figures are (A, l'') and (B, l') and whose cross-ratios are k and k_1 respectively. Let l'' and l' intersect in C, and let the join of A and B be l. S produces along the lines l and l' one-dimensional projective transformations whose cross-ratios are each k reckoned from A to B and C. (By this we mean that, if P and P_1 are two corresponding points on l, the cross-ratio k is that of the points taken in the order $(ABPP_1)$). S_1 produces along l and l'' one-dimensional projective transformations whose cross-ratios are each equal to k_1 reckoned from B to C and A. Thus along l we have the resultant of two one-dimensional transformations whose cross-ratios reckoned from A to B are k and $1/k_1$; hence, the cross-ratio of their resultant is k/k_1. The three cross-ratios taken all in the same order around the invariant triangle (ABC) are k/k_1, k_1 and $1/k$ and their product is unity.

Thus the resultant of two perspective collineations in this relative position is a non-perspective collineation which we shall designate by T. The fundamental properties of T are as follows: Its invariant figure is a triangle (ABC); along each side of the invariant triangle there is a one-dimensional transformation, and the product of the cross-ratios of these three transformations is unity when they are reckoned in the same order around the triangle; there is also a one-dimensional transformation of the pencils of rays through each vertex of (ABC), and the cross-ratios of these are the same as along the opposite sides. A collineation T is completely determined by its invariant triangle and its three cross-ratios.

Since the product of these three cross-ratios is unity, only two of them are independent. These two independent parameters may assume all possible values, and hence we see that there are ∞^2 different collineations of this type having the same invariant triangle (ABC). The resultant of any two of these collineations is evidently another of the same set, and hence the set forms a two-parameter group, which will be designated by $G_2(ABC)$.

THEOREM 13.—*There are ∞^2 collineations T, leaving one and the same triangle invariant; these form a two-parameter group $G_2(ABC)$; each collineation of this group is determined by the cross-ratios along the sides of the invariant triangle.*

21. *One-Parameter Subgroups of $G_2(ABC)$.* We shall now show that the two-parameter group $G_2(ABC)$ contains ∞^1 one-parameter subgroups. Let the

three cross-ratios along the sides AB, BC, and CA be k, k_1 and k_2; replace k_1 by k^{-r} and k_2 by k^{r-1}. The condition that the product of these cross-ratios around the triangle is unity is evidently fulfilled. A collineation T is now characterized by two constants k and r. Take from the group $G_2(ABC)$ two collineations $T(k, r)$ and $T_1(k_1, r_1)$ and form their resultant $T_2(k_2, r_2)$. Along the side AB we have $k_2 = kk_1$; along BC we have $k_2^{-r_2} = k^{-r} k_1^{-r_1}$; along CA we have $k_2^{r_2-1} = k^{r-1} k_1^{r_1-1}$.

If, now, T and T_1 be so chosen that $r_1 = r$, we see that also $r_2 = r$. Thus $k_2 = kk_1$, $k_2^{-r} = k^{-r} k_1^{-r}$ and $k_2^{r-1} = k^{r-1} k_1^{r-1}$. Hence, the resultant of two collineations in $G_2(ABC)$, for which the r's are the same, is a collineation with the same value of r. If r be kept constant and k made to vary, the resulting collineations have the group property and form a one-parameter subgroup of $G_2(ABC)$. There is a one-parameter subgroup of $G_2(ABC)$ for every value of r; thus the group $G_2(ABC)$ contains ∞^1 one-parameter subgroups, one of which is designated by $G_1(ABC)r$.

THEOREM 14.—*The two-parameter group $G_2(ABC)$ contains ∞^1 one-parameter subgroups $G_1(ABC)r$; each subgroup is characterized by a constant r, and its variable parameter is k.*

22. *The Path-Curves of $G_1(ABC)r$.* The analytic expressions for a one-parameter group $G_1(ABC)r$ can readily be written down. Let the triangle of reference be the invariant triangle (ABC), and let T be the collineation which transforms the point $P = (x, y, z)$ to $P_1 = (x_1, y_1, z_1)$. T is characterized by k and r; the cross-ratio of the pencil $C(ABPP_1)$ is k.

In terms of the coordinates of P and P_1 we have

$$\frac{z_1}{x_1} = k\frac{z}{x}\,;\quad \frac{x_1}{y_1} = k^{-r}\frac{x}{y}\,;\quad \frac{y_1}{z_1} = k^{r-1}\frac{y}{z}\,.$$

The effect of all the transformations of the one-parameter group $G_1(ABC)r$ upon a point P is to move it along a curve which is called its *path-curve.* There are ∞^1 of these path-curves so situated that every point of the plane lies on one, and only one, of these curves. The equation of this family of path-curves is found by eliminating k from the above equations; this gives

$$x^{r-1}yz^{-r} = x_1^{r-1}y_1z_1^{-r} = C, \text{ or } x^{r-1}y = Cz^r.$$

Thus the path-curves of the group $G_1\,(ABC)\,r$ are curves of order r and form a family whose parameter is C.

THEOREM 15.—*When the invariant triangle is taken as the triangle of reference, the family of path-curves of a one-parameter group* $G_1\,(ABC)\,r$ *is given by* $x^{r-1}\,y = Cz^r$, *where* r *is a constant.*

23. *Geometric Meaning of* r. It is not difficult to determine, from the equation of the family of path-curves, the geometric meaning of r. Take any point $P = (x_1, y_1, z_1)$ and draw a tangent at P to the path-curve through P and join P to the vertices of the triangle (ABC). The equation of the tangent to the path-curve at P is

$$PT = x\,(r-1)\,y_1z_1 + yx_1z_1 - rzx_1y_1 = 0.$$

The lines PA, PB, and PC are given by the equations

$$\begin{aligned} PA &= yx_1 - xy_1 = 0, \\ PB &= yz_1 - zy_1 = 0, \\ PC &= xz_1 - zx_1 = 0. \end{aligned}$$

The cross-ratio of the pencil $P\,(BATC)$ is found to be r. It is evident that this cross-ratio r is a constant for every point on every path-curve of the group $G_1\,(ABC)\,r$.

THEOREM 16.—*The constant* r *in the one-parameter group* $G_1\,(ABC)\,r$ *is one of the cross-ratios of the pencil of four lines formed by a tangent to one of the path-curves and the three lines drawn from its point of contact to the vertices of the invariant triangle.*

Having the equation of the family of path-curves given in the form $x^{r-1}y = Cz^r$, their chief properties may readily be deduced. The curves will be algebraic or transcendental, according as r is a rational or irrational number. The vertices of the invariant triangle are singular points on all curves of the family, when r is irrational; one vertex is a singular point when r is rational, except when the path-curves are conics.

24. *Path-Curves when* $r = 1, 0, \infty$. The path-curves of a one-parameter subgroup of $G_2\,(ABC)$ reduce to straight lines for three values of r, viz. $r = (1, 0, \infty)$. Let $r = 1$ in $x^{r-1}\,y = Cz^r$, and we get a pencil of lines $y = Cz$ through B; let $r = 0$ and we get $y = Cx$, a pencil of lines through A; let $r = \infty$ and we get $x = Cz$, a pencil of lines through C.

The three cross-ratios around the sides of the invariant triangle are respectively k, k^{-r}, k^{r-1}, starting from AB. When $r=1$, these become k, k^{-1}, 1; thus the one-dimensional transformation along CA is the identical transformation and every point on CA is an invariant point. Consequently, every line of the pencil through B is an invariant line. Hence, the subgroup of $G_2\,(ABC)$ for $r=1$ is a group of perspective collineations. In like manner we see that when $r=0$, we have a subgroup of perspective collineations with vertex at A and axis l''; when $r=\infty$, we have a third subgroup of perspective collineations.

Take any two of these perspective groups, as for example $H_1\,(A,\,l'')$ and $H_1\,(B,\,l')$; there are ∞^1 collineations in each group, and hence there are ∞^2 resultants obtained by combining each collineation of the one group with every one of the other. These ∞^2 resultants constitute the group $G_2\,(ABC)$; in fact, every collineation in $G_2\,(ABC)$ is the resultant of two perspective collineations, one from each of the groups $H_1\,(A,\,l'')$ and $H_1\,(B,\,l')$. Since there are three such subgroups of perspective collineations in $G_2\,(ABC)$, viz. $H_1\,(A,\,l'')$, $H_1\,(B,\,l')$ and $H_1\,(C,\,l)$, it follows that every collineation in $G_2\,(ABC)$ is in three different ways the resultant of two perspective collineations.

THEOREM 17.—*The group $G_2\,(ABC)$ contains three subgroups of perspective collineations, viz. the subgroups for which $r=1,\,0,\,\infty$. Every collineation in $G_2\,(ABC)$ is in three ways the resultant of two perspective collineations of the group.*

25. *Path-Curves when $r=-1,\,2,\,1/2$.* There are three other specially important subgroups of $G_2\,(ABC)$; these correspond to the values $r=-1,\,2,\,1/2$. Putting $r=-1$ in $x^{r-1}\,y=Cz^r$, we get $yz=Cx^2$; the path-curves reduce in this case to a system of conics having double contact at A and C. BA and BC are common tangents to the pencil of conics and AC is the chord of contact. In like manner, when $r=2$, the path-curves are $xy=Cz^2$; when $r=1/2$, the path-curves are $xz=Cy^2$. These are also pencils of conics having double contact; in each case two sides of the invariant triangle are common tangents and the third side is the chord of contact.

THEOREM 18.—*The group $G_2\,(ABC)$ contains three special subgroups for which the path-curves are conics having double contact; these are the subgroups for which $r=-1,\,2,\,1/2$.*

For any rational value of r, except the six special values $1,\,0,\,\infty,\,-1,\,2,\,1/2$, there are six subgroups for which the path-curves are algebraic curves of the same order, viz. for r, $1/r$, $1-r$, $1/(1-r)$, $(r-1)/r$, $r/(r-1)$. Thus for $r=3$ the path-curves are cubics, viz. $x^2y=Cz^3$; the path-curves are also cubics for $r=1/3$, -2, $-1/2$, $2/3$, $3/2$, and for no other values of r.

26. *Properties of the Group* $G_1(ABC)\,r$. Let the collineation T be a complex collineation, i. e. let the invariant points (ABC) be real or imaginary, and let the cross-ratios k, k_1, k_2 be complex numbers. The two parameters of the group $G_2(ABC)$ may be conveniently taken to be k and k', where k is the cross-ratio along AB and k' is that along AC. Then the cross-ratios of all collineations which belong to the subgroup $G_1(ABC)\,r$ satisfy the relation $k'=k^{1-r}$, the constant r is, in general, a complex number.

The independent parameter k of $G_1(ABC)\,r$ takes on all complex values, so that this group has many of the properties of the one-parameter group of perspective collineations. The identical collineation of the group $G_2(ABC)$ is given by $k=1$ and $k'=1$; this is contained in every one-parameter subgroup $G_1(ABC)\,r$. The group $G_1(ABC)\,r$ contains ∞^1 infinitesimal collineations from which the group may be generated. For $k=0$ and $k=\infty$ the group has two pseudo-collineations; it is continuous everywhere except at the points $k=0$ and $k=\infty$.

The subgroups of $G_2(ABC)$ for which r is rational (and hence real) have an important property which is not possessed by the groups for which r is irrational. To exhibit this property we proceed to find all collineations common to any two groups $G_1(ABC)\,r$ and $G_1(ABC)\,r'$. This is equivalent to finding all the roots common to the two equations $k'=k^{1-r}$ and $k'=k^{1-r'}$. Besides the identical and pseudo-collineations, these two groups have in common all collineations corresponding to values of k which satisfy the equation $k^{r-r'}=1$. If r is a rational number, $1-r$ is also rational; let even and odd integers be represented by e and o respectively, then we have three cases to be considered, viz. $1-r=e/o$, o/o, or o/e. In the first case, the equation $k'=k^{1-r}$ is satisfied by the pair of values $(-1, 1)$; in the second case, by the pair $(-1, -1)$; in the third case, by the pair $(1, -1)$. The collineation corresponding to the values $k=-1$ and $k'=1$ is the involutoric perspective collineation belonging to the group $H_1(B, l')$. This involutoric collineation belongs to all subgroups of $G_2(ABC)$ for which r is rational with odd numerator and odd denominator. In like manner, the collineation corresponding to the pair of values $(-1, -1)$ is the involutoric perspective collineation belonging to the group $H_1(A, l'')$. This collineation belongs to all subgroups of $G_2(ABC)$ for which r is rational with even numerator and odd denominator. The collineation corresponding to the pair of values $(1, -1)$ is the involutoric perspective collineation of the group $H_1(C, l)$; this belongs to all subgroups of $G_2(ABC)$ for which r is rational with odd numerator and even denominator. None of these three involutoric perspective collineations can belong to a subgroup of $G_2(ABC)$ for which r is irrational.

THEOREM 19.—*Every one-parameter subgroup of $G_2(ABC)$ for which r is a rational number contains an involutoric perspective collineation.*

27. *Number of Collineations of Type I.* We have seen that every collineation of type I leaves a triangle invariant and is further characterized by two independent cross-ratios k and k'. Every collineation of type I depends therefore upon eight parameters, viz. the six coordinates of the three vertices of the invariant triangle and these two independent cross-ratios. Since each of these parameters may assume ∞^1 different values, we see that there are ∞^8 collineations of type I in the plane. We are also enabled to distinguish two distinct kinds of variable parameters, viz. coordinates of invariant points and characteristic cross-ratios.

Since there are ∞^6 triangles in the plane, it follows that there are ∞^6 such two-parameter groups as $G_2(ABC)$ in the plane. Hence, the group of all collineations in the plane G_8 contains ∞^6 two-parameter subgroups $G_2(ABC)$. No two of these two-parameter groups can have a collineation of type I in common; for, if two collineations T and T_1 are identical, the eight parameters of the one must be equal to the eight parameters of the other. Now, the two cross-ratio parameters of the one may readily be equal to those of the other; but if the collineations leave different triangles invariant, all of the six coordinate parameters of the one cannot be equal to those of the other. Hence, T and T_1 cannot be identical.

28. *Three Classes of Groups of Type I.* Having now established the existence of two-parameter groups of the kind $G_2(ABC)$ and their one-parameter subgroups $G_1(ABC)\,r$, we shall proceed in the following paragraphs to enumerate the groups of higher orders which can be compounded out of these one- and two-parameter groups. In so doing, we shall find three distinct classes of such groups, viz. (1) those which are made up of two-parameter groups of the kind $G_2(ABC)$; (2) another class of groups in which every collineation is character-

ized by the same value of r; the groups of this class are made up of one-parameter groups of the kind $G_1(ABC)\,r$ in which r is a constant; (3) finally, a class of groups made up of one-parameter groups $G_1(ABC)\,r$, whose path-curves are conics, i. e. where $r = -1,\ 2,\ 1/2$. Strictly speaking, the third class is only a special case of the second class, but the special case is so important that these groups should be classified by themselves.

B.—Groups of the First Class.

By the aid of the principle that all collineations which leave a certain figure invariant, form a group, it is not difficult to enumerate all of the subgroups of the general projective group G_8 that can be compounded out of the ∞^6 two-parameter groups $G_2(ABC)$ in the plane.

29. *The Groups* $G_6(l)$ *and* $G_6(A)$. There are ∞^8 collineations of type I in the plane and only ∞^2 lines in the plane; hence, any line of the plane can be transformed into any other line or into itself in ∞^6 different ways. The ∞^6 collineations which transform a line l into itself, form a six-parameter group $G_6(l)$. There are ∞^4 triangles having the side l in common; each of these triangles is the invariant triangle of a two-parameter group $G_2(ABC)$. Thus we see that $G_6(l)$ is made up of ∞^4 two-parameter groups $G_2(ABC)$.

In like manner, we see that any point A of the plane is the invariant figure of a six-parameter group $G_6(A)$. Each of the ∞^4 triangles having A for one vertex is the invariant triangle of a two-parameter subgroup of $G_6(A)$. The groups $G_6(l)$ and $G_6(A)$ are dualistic groups. G_8 contains ∞^2 subgroups $G_6(l)$ and also ∞^2 subgroups $G_6(A)$.

Each of these groups contains, besides collineations of type I, also collineations of other types; but the collineations composing a group $G_6(l)$ or $G(A)$ are chiefly of type I. After all the types of collineations have been investigated, the structure of all these groups enumerated under type I will be examined in detail. (See §5.)

30. *The Group* $G_5(Al)$. The general projective group G_8 contains ∞^3 subgroups $G_5(Al)$, one for each lineal element in the plane. Each group $G_5(Al)$ is made up of ∞^3 two-parameter groups $G_2(ABC)$ whose invariant triangles have one vertex and one side in common. Every group $G_6(l)$ contains ∞^1 subgroups

$G_5(Al)$, one for each point on l; likewise each group $G_6(A)$ contains ∞^1 subgroups $G_5(Al)$, one for each line through A. The group $G_5(Al)$ is self-dualistic.

31. *The Groups* $G_4(AB)$ *and* $G_4(ll')$. The points AB and their join l form the invariant figure of a four-parameter group $G_4(AB)$. The group G_8 contains ∞^4 equivalent subgroups $G_4(AB)$, one for each pair of points in the plane. A group $G_4(AB)$ is composed of ∞^2 two-parameter groups $G_2(ABC)$ whose invariant triangles have the vertices A and B in common.

In like manner, G_8 contains ∞^4 equivalent subgroups $G_4(ll')$, one for each pair of lines in the plane. A group $G_4(ll')$ is composed of ∞^2 two-parameter groups $G_2(ABC)$ whose invariant triangles have one vertex and two sides in common. The groups $G_4(AB)$ and $G_4(ll')$ are dualistic. $G_6(l)$ contains ∞^2 subgroups $G_4(AB)$, also ∞^2 subgroups $G_4(ll')$; $G_6(A)$ breaks up in a similar manner.

32. *The Group* $G_4(A, l'')$. There are ∞^4 combinations of point and line in the plane where the point is not on the line. Each combination of point and line is the invariant figure of a four-parameter group $G_4(A, l'')$. The group $G_4(A, l'')$ is composed of ∞^2 two-parameter groups $G_2(ABC)$ whose invariant triangles have a vertex and the opposite side in common. Thus we see that G_8 contains ∞^4 equivalent four-parameter subgroups $G_4(A, l'')$; these groups are self-dualistic.

33. *The Group* $G_3(ABl')$. Triangles ∞^1 in number can be arranged with two of their vertices at A and B and their third vertex on a line Al'. Each of these triangles is the invariant triangle of a two-parameter group $G_2(ABC)$; these ∞^1 two-parameter groups unite to form a three-parameter group $G_3(ABl')$.

The group of all collineations in the plane G_8 contains ∞^5 equivalent subgroups $G_3(ABl')$; a group of this kind is self-dualistic. The group $G_4(AB)$ contains ∞^1 subgroups of the kind $G_3(ABl')$, one for each line through A; also one such group for each line through B. The group $G_4(ll')$ contains ∞^1 subgroups of the kind $G_3(ABl')$, one for each point on l', and also one for each point on l.

THEOREM 20.—*The group of all collineations in the plane* G_8 *contains eight subgroups which are made up of two-parameter groups of the kind* $G_2(ABC)$; *these are* $G_6(A)$, $G_6(l)$; $G_5(Al)$; $G_4(A, l'')$, $G_4(AB)$, $G_4(ll')$; $G_3(ABl')$; $G_2(ABC)$.

C.—Groups of the Second Class.

34. *Groups in which r is Constant.* We now proceed to the determination of groups of another variety, viz. those in which every collineation is characterized by the same value of r. It has already been shown that the two-parameter group $G_2(ABC)$ breaks up into ∞^1 one-parameter subgroups, each subgroup being characterized by a constant value of r. If we take all types of groups of the first class and in these keep r constant and let the other parameters vary, we will sometimes find subgroups by this process and sometimes not. The groups G_8; $G_6(l)$, $G_6(A)$; $G_5(Al)$; $G_4(A, l')$, $G_4(AB)$, $G_4(ll')$; $G_3(ABl')$ must each be examined separately.

35. *The Group $G_2(ABl')\,r$.* The three-parameter group $G_3(ABl')$ is composed of ∞^1 two-parameter groups $G_2(ABC)$; for we can construct an infinity number of triangles having two of their vertices at A and B and the third vertex C on the line l'. Choose from each of these two-parameter groups the one parameter subgroup characterized by a certain value of r. It can be shown that the aggregate of all collineations belonging to these one-parameter groups characterized by the same constant r forms a two-parameter group. Let us take two collineations T and T_1, one from each of the groups $G_2(ABC)$ and $G_2(ABC_1)$, and find their resultant. The characteristic cross-ratios of T are k, k^{-r} and k^{r-1} along the sides AB, BC, and CA respectively; those of T_1 are k_1, k_1^{-r} and $k_1^{(r-1)}$ along the sides AB, BC_1, and C_1A respectively. The resultant of $T(ABC)$ and $T_1(ABC_1)$ is $T_2(ABC_2)$, where C_2 is some point on the line ACC_1 or l'. Along AB we have $k_2 = kk_1$ for the two component one-dimensional transformations have both invariant points in common. Along ACC_1 we have $k^{r-1}k_1^{r-1} = (kk_1)^{r-1} = k_2^{r-1}$; for the two one-dimensional transformations have a common invariant point A. The two one-dimensional transformations of the pencil through the point A have both invariant rays in common, and are characterized by k^{-r} and k_1^{-r} respectively, hence their resultant is given by $k_2^{-r} = (kk_1)^{-r}$ and hence the transformation along BC_2 is also given by k_2^{-r}. Therefore, the three characteristic cross-ratios of T_2 are k_2, k_2^{-r}, and k_2^{r-1}.

Since these two collineations T and T_1, from different one-parameter subgroups but characterized by the same value of r, compound into a collineation belonging to another one-parameter group but with the same value of r, it follows that all the collineations belonging to these one-parameter groups with the same r form a two-parameter group $G_2(ABl')r$. There is one such two-parameter group within $G_3(ABl')$ for every value of r.

THEOREM 21.—*The group $G_3(ABl')$ contains ∞^1 two-parameter subgroups $G_2(ABl')r$, one for each value of r. The groups $G_3(ABl')$ and $G_2(ABl')r$ have the same invariant figure.*

36. *The Groups $G_3(ll')r$ and $G_3(AB)r$.* In like manner it can be shown that the four-parameter group $G_4(ll')$ can be split up into ∞^1 three-parameter groups such that every collineation belonging to one of these three-parameter subgroups is characterized by the same value of r. Consider ∞^2 triangles having common the sides l and l' and the vertex A. Each of these triangles determines a two-parameter group $G_2(ABC)$; take from each of these two-parameter groups the one-parameter group characterized by a certain value of r. These ∞^2 one-parameter groups form a three-parameter group every collineation in which is characterized by the same value of r. To prove this, take two collineations T from $G_2(ABC)$ and T_1 from $G_2(AB_1C_1)$. Let the three cross-ratios of T be k, k^{-r}, k^{r-1} and those of T_1 be k_1, k_1^{-r}, k^{r-1}. The resultant of T and T_1 is T_2. Along ABB_1 we have $k_2 = kk_1$, for the two one-dimensional transformations have the invariant point A in common; along ACC_1 we have $k_2^{r-1} = k^{r-1}k_1^{r-1}$ for the same reason. In the pencil through A we have $k_2^{-r} = k^{-r}k_1^{-r}$, for the two one-dimensional transformations have both invariant rays in common. Hence, the resultant T_2 has for its three cross-ratios k_2, k_2^{-r}, k_2^{r-1}.

Since the resultant of any two collineations of $G_4(ll')$ which are characterized by the same r is also characterized by the same r, it follows that all collineations in $G_4(ll')$ which have the same value of r form a subgroup $G_3(ll')r$. There is one such subgroup in $G_4(ll')$ for every value of r. The four variable parameters of $G_4(ll')$ are the segments AB and AC, k and r; if r be kept constant and the other three parameters be made to vary, we obtain the subgroup $G_3(ll')r$.

In a similar manner it may be shown that the group $G_4(AB)$ contains ∞_1 three-parameter subgroups each characterized by a constant value of r. To prove this, consider ∞^2 triangles having the two vertices in common and take the collineations $T(ABC)r$ and $T_1(ABC_1)r$; their resultant is $T_2(ABC_2)r$. The details of the proof are left for the reader to supply. The two groups $G_3(ll')r$ and $G_3(AB)r$ are dualistic to each other.

THEOREM 22.—*The group* $\left\{\begin{matrix} G_4(ll') \\ G_4(AB) \end{matrix}\right\}$ *contains* ∞^1 *subgroups* $\left\{\begin{matrix} G_3(ll')\,r \\ G_3(AB)\,r \end{matrix}\right\}$, *one for each value of* r.

37. *The Group* $G_4(Al)\,r$. The five parameter group $G_5\,(Al)$, whose invariant figure is the lineal element Al contains ∞^1 four-parameter subgroups each characterized by a constant value of r. To show this, take two collineations $T(ABC)\,r$ and $T_1\,(AB_1C_1)\,r$, having the same value of r, from the two-parameter groups $G_2\,(ABC)$ and $G_2\,(AB_1C_1)$ whose invariant triangles have the vertex A and the side l in common. The cross-ratios of T are k, k^{-r}, k^{r-1}; those of T_1 are k_1, k_1^{-r}, k_1^{r-1}. Along ABB_1 we have $k_2 = kk_1$, for these two one-dimensional transformations have the invariant point A in common. In the pencil through A we have $k_2^{-r} = k^{-r}\,k_1^{-r}$, for these two one-dimensional transformations have the ray ABB_1 in common. The third cross-ratio of the resultant T_2 must be k_2^{r-1}, for the product of the three cross-ratios taken in order around the triangle must be unity. The three cross-ratios of T_2 are, therefore, k_2, k_2^{-r}, k_2^{r-1} and its invariant triangle is AB_2C_2 where B_2 is on the line ABB_1. Thus the resultant of any two collineations in $G_5\,(Al)$, both characterized by the same value of r, is also characterized by the same r. Thus we see that the group $G_5\,(Al)$ contains ∞^1 subgroups $G_4\,(Al)\,r$, one for each value of r.

THEOREM 23.—*The group* $G_5\,(Al)$ *contains* ∞^1 *four-parameter subgroups* $G_4\,(Al)\,r$, *one for each value of* r. *These subgroups all have in common the same invariant lineal element.*

38. *Subgroups of* $G_4\,(A,\,l'')$, $G_6\,(l)$, $G_6\,(A)$, G_8. It is easy to show that the remaining groups of the first class, viz. $G_4(A,\,l'')$, $G_6(l)$, $G_6\,(A)$, and G_8 do not break up into subgroups in the above manner. For example, combine two collineations T and T_1, characterized by the same value of r, from $G_6\,(l)$; the invariant triangles of T and T_1 have the line l for a common side, but no common vertex. Along the invariant side l the law of combination $k_2 = kk_1$ does not hold, because the two one-dimensional transformations have no invariant point in common; hence, there is no reason why the resultant T_2 should be characterized by the same r as T and T_1. Like reasons hold in the case of the other three groups $G_4\,(A\,,\,l'')$, $G_6\,(A)$ and G_8; thus, in general, the three groups $G_4\,(A,\,l'')$, $G_6\,(A)$, and $G_6\,(l)$ do not break up into subgroups characterized by a constant r. There is, however,

one exceptional value of r for which these three groups each yield one subgroup. This exceptional case will be discussed later. See articles 44 and 45.

39. *Critical Values of r.* The six critical values of r, viz. $r = 1, 0, \infty, -1, 2, 1/2$ are to be excepted for each of these groups of the second class. For $r = 1, 0, \infty$ the collineations are all perspective collineations and the groups corresponding to these values of r are groups of perspective collineations. For $r = -1, 2, 1/2$ the path-curves are conics and the resulting groups belong to the third class to be considered below.

THEOREM 24.—*There are five, and only five, varieties of groups of the second class, viz.* $G_1(ABC)\,r$, $G_2(ABl')\,r$, $G_3(AB)\,r$, $G_3(ll')\,r$, $G_4(Al)\,r$; *of these the two groups* $G_3(AB)\,r$ *and* $G_3(ll')\,r$ *are dualistic and the other three are self-dualistic.*

D.—Groups of the Third Class.

40. *Groups whose Path-Curves are Conics.* We come now to the consideration of the groups of collineations of type I which are made up of one-parameter groups whose path-curves are conics. The groups which we shall find are, in many instances, only special cases of the groups $G_2(ABl')\,r$, $G_3(AB)\,r$, $G_3(ll')\,r$ and $G_4(Al)\,r$ when $r = -1, 2, 1/2$; but we shall find many groups which are not special cases of the above, and hence it is best, even at the risk of being tedious, to enumerate all varieties of groups of this class.

There is only one variety of one-parameter group of this kind; in this case the two common tangents to the conics and the common chord of contact form the invariant triangle. This group will be symbolized by $G_1(K_n)$.

41. *The Groups $G_3(K)$ and $G_2(AKl)$.* A very important group $G_3(K)$, with invariant conic K, can be built up as follows: ∞^2 triangles can be constructed consisting of two tangents to K and their chord of contact. Belonging to each such triangle is a one-parameter group $G_1(K_n)$ having K among its system of invariant conics. These ∞^3 collineations, leaving K invariant, form a three-parameter group $G_3(K)$ by the principle that the aggregate of all collineations, leaving a given figure invariant, forms a group.

The group $G_3(K)$ contains ∞^1 two-parameter subgroups which may be constructed as follows: take a conic K and a tangent l to it at a point A of the conic. We can construct ∞^1 triangles having one vertex at A, another vertex at B on l, and a third vertex C on the conic such that AB and BC are tangents to K, and AC is the chord of contact. All one-parameter groups belonging to these triangles, whose path-curves are conics touching AB at A, have K among their invariant path-curves. Hence, these collineations, leaving invariant K, a point on K, and the tangent to K at the point, form a two-parameter group $G_2(AKl)$. The two variable parameters of this group are the cross-ratio k and the position of the point C on the conic K.

THEOREM 25.—*All collineations of the plane which leave a conic K invariant, form a three-parameter group $G_3(K)$; this group contains ∞^1 two-parameter subgroups $G_2(AKl)$, one for each point on K.*

The groups $G_2(ABl')\,r$, $G_3(AB)\,r$, $G_3(ll')\,r$ and $G_4(Al)\,r$, where $r = -1$, 2, 1/2 are all special cases of those of the second class. Three of these groups of the same order differ only in the arrangement of the invariant triangles of the one-parameter subgroups $G_1(K_n)$.

42. *The Group $G_3(S)$.* There is still another three-parameter group of collineations composed of one-parameter groups $G_1(K_n)$. Take a lineal element Al and a pencil of conics S all touching l at A and having at A contact of the third order with one another. One property of this pencil of conics is that the polars of a point on l with respect to the pencil S coincide, and this common polar passes through A. A pencil of conics having contact of the third order is projectively transformed into a pencil of the same kind. We can construct ∞^2 triangles having one line AB along l, one vertex at A, and one vertex at C on one of these conics, so that BC is a tangent to the curve at C, and AC is the chord of contact. Belonging to each of these triangles is a one-parameter group $G_1(ABC)\,r = -1$ such that one conic of the pencil S is included in its family of path-curves. Every collineation in such a one-parameter group leaves invariant one of the conics of S and interchanges the other conics of S; hence, it leaves invariant the pencil S as a whole and the lineal element Al. The aggregate of all collineations, leaving S invariant, forms a three-parameter group $G_3(S)$.

Since the group $G_3(S)$ leaves invariant the lineal element Al, and the pencil of conics S having contact of the third order with Al, we see that $G_3(S)$ can be built up out of ∞^1 two-parameter groups $G_2(AKl)$, one for each conic in S.

Again, since the polars of any point B on l with respect to each conic of S coincide in l' a line through A, we see that $G_3(S)$ is composed of ∞^1 subgroups $G_2(ABl')\, r = 1$.

43. *The Structure of* $G_4(Al)\, r = -1$. The group $G_4(Al)\, r = -1$ breaks up into subgroups in several different ways, and accordingly its structure is peculiar and worthy of special attention.

There are ∞^3 distinct conics touching l at A. This system of conics is composed of ∞^1 nets; each net contains ∞^2 conics having contact of the second order at A, and hence all conics of the net have a common circle of curvature at A. Each of the circles touching l at A is the circle of curvature of such a net.

Each net N is composed of ∞^1 pencils S such that the conics of each pencil have contact of the third order at A. The polar of the point at infinity on l with respect to the conics of the pencil S is a line l' through A, the line of centres of S. Each line through A is the line of centres of one of the pencils of the net N. Since there are ∞^1 nets N, each line through A is the line of centres of ∞^1 pencils S. The system of conics touching l at A is, therefore, composed of ∞^2 pencils S.

Each pencil S is the invariant figure of a group $G_3(S)$, hence, there are ∞^2 such groups all contained within the group $G_4(Al)\, r = -1$. But since $G_4(Al)\, r = -1$ contains ∞^4 collineations and ∞^2 subgroups $G_3(S)$, it seems that two such subgroups as $G_3(S)$ and $G_3(S_1)$ must contain certain collineations in common. Let us consider two pencils of conics S and S_1 which do not belong to the same net N. A given conic K of the pencil S is cut by each conic of S_1 in two points other than A. When the two points of intersection of K and K_1 coincide, the conics K and K_1 have double contact. Each conic of S has double contact with one, and only one, conic of S_1. We wish to find the locus of the points of contact of corresponding conics of S and S_1.

The polars of a point B on l with respect to all conics of S is the same line l' through A. Thus the range of points on l and the pencil of lines through A are projectively related by means of S. In the same manner, by means of S_1, the range of points on l is projectively related to the pencil of lines through A. These two pencils of lines being projective to one another, have two self-corresponding rays. One of these is l, since l is the polar of A with respect to both S and S_1. The other self-corresponding ray is a line l' through A. Hence, the polars of a certain point B on l, with respect to both S and S_1, coincide in l', and the locus of the points of contact of conics of S and S_1 is l', a straight line through A. Therefore, we see that the two three-parameter groups $G_3(S)$ and $G_3(S_1)$ contain the same two-parameter group $G_2(ABl')\,r = -1$.

On the other hand, if the two pencils S and S_1 belong to the same net N of conics having contact of the second order at A, the conics from S and S_1 cannot have double contact, and hence two such groups $G_3(S)$ and $G_3(S_1)$ have no collineations in common.

In a certain net N there are ∞^1 pencils S, S_1, S_2, etc.; each of these pencils is the invariant pencil of a group $G_3(S_n)$. ∞^1 such groups $G_3(S_n)$, no two of which contain a collineation of type I in common, include all collineations of this type in $G_4(Al)\,r = -1$. It is easy to see that the ∞^3 subgroups $G_1(ABC)\,r = -1$ in $G_4(Al)\,r = -1$ are all included in the ∞^1 groups $G_3(S_n)$ of the net N. If it be true that all collineations of type I in $G_3(Al)\,r = -1$ are included in the ∞^1 groups $G_3(S_n)$ of the net N, then all collineations of type I in a group $G_3(S')$, where S' is a pencil of conics not included in N, are to be found in the ∞^1 groups $G_3(S_n)$ of the net N. In fact, if we take $G_3(S')$ in turn with each of the groups $G_3(S_n)$ of the net, we see that $G_3(S')$ has a two-parameter subgroup $G_2(ABl')\,r = -1$ in common with each group $G_3(S_n)$ of the net; the common subgroup $G_2(ABl')\,r = -1$ is different for each of the groups $G_3(S_n)$ of the net. In this way it can be shown without difficulty that every collineation of type I in $G_3(S')$ is also to be found in the net of groups $G_3(S_n)$.

THEOREM 26.—*The group $G_4(Al)\,r = -1$ contains ∞^2 subgroups $G_3(S)$; two subgroups $G_3(S)$ and $G_3(S_1)$ contain no collineation of type I in common when the two pencils are S and S_1 belong to the same net N; they have a subgroup $G_2(ABl)\,r = -1$ in common when S and S_1 belong to different nets. Each group $G_3(S)$ contains ∞^1 subgroups $G_2(AKl)$, one for each conic in S.*

44. *The Group $G_3(A, l'')\,r = 2$.* It was remarked in article 38 that the group $G_4(A, l'')$ does not break up into subgroups $G_3(A, l'')\,r$ for all values of r; but that one such subgroup exists, viz. $G_3(A, l'')\,r = 2$. In order to prove this, we take two collineations $T(ABC)\,r$ and $T_1(AB_1C_1)\,r$ characterized by the same value of r. Their invariant triangles (ABC) and (AB_1C_1) have the vertex A and the side l'' opposite A in common. The cross-ratios of T are k, k^{-r}, k^{r-1} along AB, BC and CA respectively; the cross-ratios of T_1 are k_1, k_1^{-r} and k_1^{r-1}

along AB_1, B_1C_1, C_1A respectively. The resultant of T and T_1 is T_2 and leaves invariant a triangle AB_2C_2, where B_2 and C_2 are some pair of points on the line BC or l''.

Introduce now an auxiliary collineation U, leaving invariant the triangle ABB_1 and characterized by λ and r. We can show that the value of λ may be so chosen that the resultant of T and U leaves invariant the triangle ABB_2, while the resultant of U^{-1} and T_1 leaves invariant the triangle AB_1C_2.

Since the invariant triangles of T and U are respectively ABC and ABB_1, the resultant TU of T and U will leave invariant a triangle ABX, where X is some point on l''. Since the invariant triangles of U^{-1} and T_1 are respectively AB_1B and AB_1C_1, the resultant $U^{-1}T$ of U^{-1} and T will leave invariant a triangle AB_1X_1, where X_1 is some point on l''. If λ be made to vary through all values, the points X and X_1 will each assume in turn all positions on the line l''. The two projective ranges, thus generated by X and X_1, will have two self-corresponding or double points, D and D_1, in common. Let λ_1 be that value of λ for which X and X_1 coincide in D, and let U be so chosen that its characteristic constants are λ_1 and r. Since the invariant triangles of TU and $U^{-1}T_1$ are now AD_1B and AD_1B_1, the invariant triangle of their resultant will be AD_1D'. But this resultant is $TUU^{-1}T = TT_1$ whose invariant triangle is AB_2C_2. Hence, the point D' is either B_2 or C_2. In like manner, by choosing for λ a value λ_2 for which X and X_1 coincide at D_1, it may be shown that D_1 is either B_2 or C_2. Hence, λ may always be so chosen, and in two distinct ways, so that the resultant of T and U leaves invariant the triangle ABB_2, while the resultant of U^{-1} and T_1 leaves invariant the triangle AB_1C_2.

The collineations T and T_1, U and U^{-1}, with their cross-ratios, may be written symbolically as follows:

$$T(ABC) = \left\{ \frac{k}{AB}, \frac{k^{-r}}{BC}, \frac{k^{r-1}}{CA} \right\}; \quad T_1(AB_1C_1) = \left\{ \frac{k_1}{AB_1}, \frac{k_1^{-r}}{B_1C_1}, \frac{k_1^{r-1}}{C_1A} \right\},$$

$$U(ABB_1) = \left\{ \frac{\lambda}{AB}, \frac{\lambda^{-r}}{BB_1}, \frac{\lambda^{r-1}}{B_1A} \right\}; \quad U^{-1}(ABB_1) = \left\{ \frac{\lambda^{-1}}{AB}, \frac{\lambda^{r}}{BB_1}, \frac{\lambda^{1-r}}{B_1A} \right\}$$

or

$$U^{-1}(AB_1B) = \left\{ \frac{\lambda^{r-1}}{AB_1}, \frac{\lambda^{-r}}{B_1B}, \frac{\lambda}{BA} \right\}.$$

Since the invariant triangles of T and U have two vertices in common, the parameters k and k_1 combine according to the law $kk_1 = k_2$; the same is true for

U^{-1} and T_1, and also for TU and $U^{-1}T_1$. Hence, we have

$$TU(ABB_2) = \left\{\begin{matrix} (k\lambda) \\ AB \end{matrix}, \begin{matrix} (k\lambda)^{-r} \\ BB_2 \end{matrix}, \begin{matrix} (k\lambda)^{r-1} \\ B_2A \end{matrix}\right\}$$

or

$$TU(AB_2B) = \left\{\begin{matrix} (k\lambda)^{1-r} \\ AB_2 \end{matrix}, \begin{matrix} (k\lambda)^{r} \\ B_2B \end{matrix}, \begin{matrix} (k\lambda)^{-1} \\ BA \end{matrix}\right\}$$

and

$$U^{-1}T_1(AB_1B_2) = \left\{\begin{matrix} (k_1\lambda^{r-1}) \\ AB_1 \end{matrix}, \begin{matrix} (k_1\lambda)^{-r} \\ B_1B_2 \end{matrix}, \begin{matrix} (k_1^{r-1}\lambda) \\ B_2A \end{matrix}\right\}$$

or

$$U^{-1}T_1(AB_2B_1) = \left\{\begin{matrix} (k_1^{1-r}\lambda^{-1}) \\ AB_2 \end{matrix}, \begin{matrix} (k_1\lambda) \\ B_2B_1 \end{matrix}, \begin{matrix} (k_1^{-1}\lambda^{1-r}) \\ B_1A \end{matrix}\right\}.$$

The resultant of TU and $U^{-1}T_1$ is $TUU^{-1}T_1 = TT_1$, and may be written

$$TT_1(AB_2C_2) = \left\{\begin{matrix} (kk_1)^{1-r}\lambda^{-r} \\ AB_2 \end{matrix}, \begin{matrix} (kk_1\lambda^2)^{r} \\ B_2C_2 \end{matrix}, \begin{matrix} (kk_1)^{-1}\lambda^{-r} \\ C_2A \end{matrix}\right\}.$$

From this we see that the collineation TT_1 is not, in general, characterized by the same r which characterizes T and T_1. For this to be true, the cross-ratios along B_2C_2 and C_2A must be equal to that along AB_2 raised to the power $-r$ and $r-1$ respectively; i. e. we must have the two equations

$$(kk_1\lambda^2)^r = [(kk_1)^{1-r}\lambda^{-r}]^{-r} \text{ and } (kk_1)^{-1}\lambda^{-r} = [(kk_1)^{1-r}\lambda^{-r}]^{r-1}.$$

All these conditions are fully satisfied when, and only when, $r^2 = 2r$; i. e. when $r = 0$ or $r = 2$. But if $r = 0$, T and T_1 are both perspective collineations, and their resultant is obviously also a perspective collineation. Hence, for collineations of type I we have only the result $r = 2$.

Therefore, the resultant of two collineations T and T_1, both taken from the group $G_4(A, l')$ and characterized by the same value of r, is not a collineation characterized by the same r except when $r = 2$. Hence, the group $G_4(A, l')$ does not break up into three-parameter subgroups, one for each value of r. But all collineations in $G_4(A, l')$, for which $r = 2$, do form a three-parameter group $G_3(A, l')\,r = 2$. In this group k_2 is not equal to the product of k and k_1.

THEOREM 27.—*The group $G_4(A, l')$ contains one, and only one, three-parameter subgroup of the form $G_3(A, l')\,r$, viz. $G_3(A, l')\,r = 2$.*

45. *The Groups $G_5(l)\,r = 2$ and $G_5(A)\,r = 2$.* We shall now show in a manner similar to the above that the group $G_6(l)$ does not break up into sub-

groups $G_5(l)r$ for all values of r, but that one such subgroup exists for $r = 2$. Take two collineations T and T_1 thus:

$$T(ABC) \equiv k,\ k^{-r},\ k^{r-1} \text{ and } T_1(A_1B_1C_1) \equiv k_1,\ k_1^{-r},\ k_1^{r-1},$$

in which the triangles ABC and $A_1B_1C_1$ have one side l' but no vertex in common; the resultant of T and T_1 is $TT_1(A_2B_2C_2)$, where B_2 and C_2 are a pair of points on l'. Introduce a collineation U and its inverse U^{-1} as follows:

$$U(OB_1C) \equiv \lambda,\ \lambda^{-r},\ \lambda^{r-1}, \quad U^{-1}(OB_1C) \equiv \lambda^{-1},\ \lambda^{r},\ \lambda^{1-r},$$

where O is the intersection of A_1B_1 and AC. The resultants of T and U, U^{-1} and T_1 are as follows:

$$TU(XB_1C) \equiv (k\lambda),\ (k\lambda)^{-r},\ (k\lambda)^{r-1};$$

$$U^{-1}T_1(X_1B_1B_2) \equiv (k_1\lambda^{-1}),\ (k_1\lambda^{-1})^{-r},\ (k_1\lambda^{-1})^{r-1}$$

or

$$U^{-1}T_1(X_1B_2B_1) \equiv (k_1\lambda^{-1})^{1-r},\ (k_1\lambda^{-1})^{r},\ (k_1\lambda^{-1})^{-1}.$$

Here λ is so chosen that the invariant triangles of TU and $U^{-1}T_1$ have one vertex B_2 in common. The possibility of this may be shown as in last article; the resultant of TU and $U^{-1}T_1$ is evidently TT_1.

In combining $TU(XB_2C)$ and $U^{-1}T_1(X_1B_2B_1)$, we see that along the side BC or l' the two one-dimensional transformations have the invariant point B_2 in common; through the point B_2 the two one-dimensional transformations of the pencil have the invariant ray l' in common. Thus, in both these cases, the law $kk_1 = k_2$ holds good. Hence, we have

$$TT_1(A_2B_2C_2) \equiv \left\{ \frac{kk_1^{1-r}\lambda^{r}}{A_2B_2},\ \frac{k^{-r}k_1^{r}\lambda^{-2r}}{B_2C_2},\ \frac{k^{r-1}k_1^{-1}\lambda_{r}}{C_2A_2} \right\}.$$

In this resultant collineation the cross-ratio along B_2C_2 is not that along A_2B_2 raised to the power $-r$ except for $r = 2$ or 0. Thus, if $(kk_1^{1-r}\lambda^{r})^{-r} = k^{-r}k_1^{r}\lambda^{-2r}$, we must have

$$k_1^{r^2-r}\lambda^{-r^2} = k_1^{r}\lambda^{-2r}.$$

This is satisfied when $r^2 = 2r$, i. e. when $r = 2$ or 0. But if $r = 0$, T and T_1 are both perspective collineations. Hence, for collineations of type I, we have only the result $r = 2$.

It may be shown in a similar manner that the resultant of two collineations T and T_1, taken from $G_6(A)$ and characterized by the same value of r, is also characterized by this value of r only when $r = 2$. The proof is left for the reader.

THEOREM 28.—*The group* $\left\{\begin{matrix} G_6(l) \\ G_6(A) \end{matrix}\right\}$ *contains one, and only one, five-parameter subgroup of the kind* $\left\{\begin{matrix} G_5(l)\,r \\ G_5(A)\,r \end{matrix}\right\}$, *viz. the group* $\left\{\begin{matrix} G_5(l)\,r = 2 \\ G_5(A)\,r = 2 \end{matrix}\right\}$.

46. *Summary of Groups of Third Class.* We have thus found eleven varieties of groups of the third class. Five of these eleven groups, viz. $G_1(ABC)\,r$, $G_2(ABl')\,r$, $G_3(AB)\,r$, $G_3(ll')\,r$, and $G_4(Al)\,r$, where $r = -1,\ 2,\ 1/2$, are only special cases of groups of the second class. On the other hand, the six varieties of groups, viz. $G_2(AKl)$, $G_3(K)$, $G_3(S)$, $G_3(A, l'')\,r = 2$, $G_5(l)\,r = 2$ and $G_5(A)\,r = 2$ are essentially of the third class.

THEOREM 29.—*There are six distinct varieties of groups of the third class, viz.* $G_2(AKl)$, $G_3(K)$, $G_3(S)$, $G_3(A, l'')\,r = 2$, $G_5(l)\,r = 2$ *and* $G_5(A)\,r = 2$.

§3.—COLLINEATIONS OF TYPE II.

A. The Group $G'_2(ABl')$ and its Subgroups.

47. *The Properties of a Collineation T'.* The kind of collineation which we shall call type II is most easily obtained as the resultant of a perspective collineation $S(B, l')$ and an elation $S'(Al)$. Let the invariant figures of S and S' be so chosen that B, the vertex of S, lies on l the axis of S', but not at A; also let l', the axis of S, pass through A the vertex of S', and let ϕ be the angle between l and l'. Let the cross-ratio of S be k and the constant of S' be α. The resultant of S and S' leaves invariant the points A and B and the lines l and l'. This resultant will be designated by T' and its invariant figure by (ABl').

Along the line l, S' produces an identical transformation i and S produces a one-dimensional transformation t, whose invariant points are A and B, and whose cross-ratio is k; their resultant is t. Along l', S produces an identical transformation and S' a one-dimensional transformation t', whose single invariant point is A, and whose constant is $\alpha \sin \phi$; their resultant is t'. Through the point A S' produces an identical transformation i, and S a one-dimensional transformation t_1, with two invariant rays l and l', and whose cross-ratio is k; their resultant is t_1. Through B, S produces an identical transformation i and S' a one-dimensional transformation t_1', with a single invariant ray l and a constant $a\alpha$, where a is the distance AB; their resultant is t_1'. Hence T' produces along AB and through A, one-dimensional transformations with two invariant elements and the same cross-ratio k, and along l' and through B, one-dimensional transformations with a single invariant element.

If the cross-ratio along AB be k, in the direction from A to B, then that through A is also k in the direction from l' to l. The relation of the constants of the transformations along l' and through B is found as follows: Let the constant along l' be α' and that through B be β; then $\alpha' = \alpha \sin \phi$ and $\beta = a\alpha$. Hence, $\alpha' = \frac{\sin \phi}{a} \beta$. Hereafter we shall designate the constant along l' by α and that through B by β, implying the relation

$$\alpha = \frac{\sin \phi}{a} \beta .$$

48. *The Group* $G_2'(ABl')$. A collineation T' is completely characterized by its invariant figure (ABl') and two independent constants k and α. It is evident that if k and α be made to vary continuously, we have ∞^2 collineations T' with the same invariant figure (ABl'). Two of these collineations, $T'(k, \alpha)$ and $T_1'(k_1, \alpha_1)$, combine to form $T_2'(k_2, \alpha_2)$, where $k_2 = kk_1$ and $\alpha_2 = \alpha + \alpha_1$. T_2' has the same invariant figure as T' and T_1'. These ∞^2 collineations, leaving (ABl') invariant, form a two-parameter group $G_2'(ABl')$, the parameters being k and α.

THEOREM 30.—*All collineations of type II, which have the same invariant figure* (ABl'), *form a two-parameter group* $G_2'(ABl')$.

49. *One-Parameter Groups* $G_1'(ABl')\,a$. We shall now go on to show that the group $G_2'(ABl')$ contains ∞^1 one-parameter subgroups. The group $G_2'(ABl')$ depends on two independent parameters, k and α. Let us assume a relation between these quantities and consider only those collineations of the group $G_2'(ABl')$ for which k and α satisfy the assumed relation. Let $k = a^\alpha$, where a is some constant.

Take two collineations, $T'(k, \alpha)$ and $T_1'(k_1, \alpha_1)$, belonging to $G_2'(ABl')$ and combine them to form $T_2'(k_2, \alpha_2)$; let $k = a^\alpha$ and $k_1 = a_1^{\alpha_1}$. We have $\alpha_2 = \alpha + \alpha_1$ and $k_2 = kk_1$, i. e. $a_2^{\alpha_2} = a^\alpha a_1^{\alpha_1} = a^{\alpha + \alpha_1} (a_1/a)^{\alpha_1}$. If we put $a_1 = a$, we get

$k_2 = a_2^{\alpha_2} = a^{\alpha_2}$; whence we see that $a_2 = a_1 = a$. Therefore two collineations T' and T'_1, both characterized by the same value of a, result is a collineation also characterized by the same a. From this we see that all collineations of the group $G'_2(ABl')$, for which the parameters k and α satisfy the relation $k = a^\alpha$, form a subgroup of $G'_2(ABl')$. This subgroup $G'_1(ABl')a$ is characterized by a constant a, and has only one parameter α; it is therefore a one-parameter subgroup of $G'_2(ABl')$. Within $G'_2(ABl')$ there are ∞^1 such subgroups, one for each value of a (except $a = 0$).

50. *The Path-Curves of* $G'_1(ABl')a$. The effect upon a point of the plane of all the collineations of the group $G'_1(ABl')a$, is to move it along its path-curve. The equation of the family of path-curves of the group $G'_1(ABl')a$ is readily obtained as follows: In the fundamental figure ABl', let the line AB be the axis of x and l' the axis of y, so that A is the origin of a system of cartesian coordinates, generally oblique. The collineation $T'(a^\alpha, \alpha)$ transforms the point (x, y) to (x_1, y_1). The cross-ratio a^α of the pencil through A is given by $y_1/x_1 = a^\alpha y/x$; the parameter of the pencil through B is given by $(x_1 - c)/y_1 = (x - c)/y + \beta$, $(\cot\theta_1 = \cot\theta + \beta)$, c being the distance AB. Eliminating α from these equations, we have, putting $b = a^{\frac{\sin\phi}{c}}$,

$$\frac{y_1/x_1}{b^{\frac{x_1 - c}{y_1}}} = \frac{y/x}{b^{\frac{x - c}{y}}} = C\text{, whence } y = Cxb^{\frac{x-c}{y}}.$$

This represents the path-curves of the group $G'_1(ABl')a$, C being the parameter of the family of curves. These curves all pass through A and B and have these points as singular points.

THEOREM 31.—*The two-parameter group* $G'_2(ABl')$ *contains* ∞^1 *one-parameter subgroups* $G'_1(ABl')a$; *each subgroup is characterized by a constant* a *and a parameter* α. *The path-curves of* $G'_1(ABl')a$ *are given by* $y = Cxa^{\frac{x-c}{y}}$

51. *Two Special Subgroups of* $G'_2(ABl')$. Among the ∞^1 subgroups of $G'_2(ABl')$, two are of particular importance, and require special attention. When $a = 1$, we have $k = 1^\alpha = 1$ for all values of α. This signifies that the one-dimensional transformations along AB and through A are both identical transfor-

mations, and hence all points on l and all lines through A are invariant under all collineations of the group. This particular subgroup of $G'_2(ABl')$ is, therefore, the group of elations $H'_1(Al)$.

When $\alpha = 0$ and k varies, this is equivalent to $a = \infty$; the one-dimensional transformations along l' and through B are both identical. In this case, all points on l' and all lines through B are invariant under all collineation of the group. This subgroup of $G'_1(ABl')$ is, therefore, the group of perspective collineations $H_1(B, l')$.

Every collineation T' in the group $G'_2(ABl')$ is the resultant of an elation from the group $H'_1(Al)$ and a perspective collineation from the group $H_1(B, l')$. In fact, if we combine every elation in $H'_1(Al)$ with every perspective collineation in $H_1(B, l')$, we obtain ∞^2 collineations T' which constitute the group $G'_2(ABl')$.

THEOREM 32.—*The two-parameter group $G'_2(ABl)$ contains one subgroup of elations $H'_1(Al)$ and one subgroup of perspective collineations $H_1(B, l')$.*

52. *Properties of the Group $G'_1(ABl')\alpha$.* The parameter of the group is α, and the law of combination of parameters is $\alpha_2 = \alpha + \alpha_1$. Consequently, the properties of the group $G'_1(ABl')\alpha$ are quite similar to those of the one-parameter group of elations. The identical collineation of the group is given by $\alpha = 0$, pseudo-collineations are given by $\alpha = \infty$ and $k = 0$. The group contains ∞^1 infinitesimal collineations corresponding to the infinitesimal values of α; hence the group is continuous everywhere except at $\alpha = \infty$. The groups $G'_1(ABl')\alpha$ and $G'_1(ABl')\alpha_1$ contain no collineations in common save the identical and the pseudo-collineations.

53. *Classification of the Groups of Type II.* We are now in position to determine the number of collineations of type T' in the plane. Since there are ∞^5 different figures of the kind (ABl') in the plane, it follows that there are ∞^7 collineations of the kind T'; but it does not follow that these form a seven-parameter group. In general, two collineations T and T'_1 having no parts of their invariant figures in common, do not result in a collineation of type II but of type I.

Our next problem is to determine all types and varieties of groups composed chiefly of collineations of type II. We shall find two distinct classes of such groups, viz. (1) groups made up of two-parameter groups $G_2'(ABl')$, and (2) groups made up of collineations, all of which are characterized by the same value of a. In groups of the first class a and α both vary, and in groups of the second class a is a constant and α a variable.

B.—Groups of Type II of the First Class.

54. *The Groups $G_3'(AB)$ and $G_3'(ll')$.* Take two groups G_2' whose invariant figures are (ABl') and (AB_1l'), the points A, B, and B_1 being collinear. Take two collineations $T'(k, \alpha)$ and $T_1'(k_1, \alpha_1)$, one from each group, and find their resultant $T_2'(k_2, \alpha_2)$. From the position of the invariant figures, we see that $k_2 = kk_1$ and $\alpha_2 = \alpha + \alpha_1$. Hence, the invariant figure of T_2' is (AB_2l'), B_2 being some point on AB, and the resultant collineation is of type II. Thus all the collineations in the two parameter groups G_2', whose invariant figures are the same, except that the point B is in different positions on the line AB, form a three-parameter group $G_3'(ll')$.

In a similar manner, it may be shown that all the collineations in the two-parameter groups G_2', whose invariant figures are the same, except that the line l' is in different positions through the point A, form a three-parameter group $G_3'(AB)$ dualistic to $G_3'(ll')$.

55. *The Group $G_4'(Al)$.* Again let us take two collineations $T'(k, \alpha)$ and $T_1'(k_1, \alpha_1)$, one from each of the groups $G_2'(ABl')$ and $G_2'(AB_1l_1')$ whose invariant figures have common the point A and the line l, but not the same line l' or the point B, and form their resultant $T_2'(k_2, \alpha_2)$. The one-dimensional transformations along AB, since they have the point A in common, combine according to the law $k_2 = kk_1$; also the one-dimensional transformations of the pencil through A, since they have the ray l in common, combine according to the law $k_2 = kk_1$. The resultant collineation, $T_2'(k_2, \alpha_2)$, is, therefore, of type II and its invariant figure is (AB_2l_2'). Hence, the collineations in the two parameter groups G_2', whose invariant figures are (AB_nl_n') (where B_n is in turn every point on l and l_n' is in turn every line through A), form a four-parameter group $G_4'(Al)$, whose invariant figure is the lineal element (Al). This group is self-dualistic for the reason that the figure (AB_nl_n) is a self-dualistic figure.

56. *The Group $G_4'(Al')$.* There are two other varieties of four-parameter groups, dualistic to each other, which can be formed from two-parameter groups

$G'_2(ABl')$. Take two groups G'_2 whose invariant figures are (ABl') and (AB_1l'), having the line l' and the point A in common; take a collineation from each group and combine them. The two one-dimensional transformations of the kind t' along l' result in a transformation t' of the same kind, leaving invariant A and l'. The two transformations of the pencil through A, since they have the ray l' in common, combine according to the law $k_2 = kk_1$, and result in a transformation of the same kind, leaving invariant A and l' and some second line l_2 through A. The principle of duality requires that the transformation along l_2 should be of the kind t with two invariant points, hence, there is a second invariant point B_2 somewhere on l_2. The resultant collineation T'_2 is, therefore, of type II and its invariant figure is (AB_2l'). Hence, the collineations contained in the ∞^2 two-parameter groups G'_2, whose invariant figures are (AB_nl), where B_n is in turn every point in the plane, form a four-parameter group $G'_4(Al')$. This group is evidently of a different variety from that last discussed.

57. *The Group of* $G'_4(Bl)$. The group dualistic to the above is composed of ∞^2 two-parameter groups G'_2 whose invariant figures have in common the point B and the line l. Take two collineations T and T'_1, one from each of the groups $G'_2(ABl')$ and $G'_2(A_1Bl'_1)$, and combine them. The two one-dimensional transformations of the kind t' of the pencil through B result in a transformation of the same kind; the two transformations of the kind t along l, since they have the point B in common, result in a transformation of the same kind according to the law $k_2 = kk_1$. The resultant of T' and T'_1 is, therefore, of type II, and all the collineations belonging to the above-mentioned two-parameter groups unite to form a four-parameter group $G'_4(Bl)$.

58. *Summary of Groups of the First Class.* We have now enumerated all groups of this class that can be compounded out of the ∞^5 two-parameter groups $G'_2(ABl)$. For example, ∞^3 two-parameter groups G'_2 can be so chosen that all collineations in them leave the line l invariant; but the resultant of any two collineations taken from different groups G'_2, which have no invariant point in common, is no longer of type II but of type I. Hence, these ∞^3 groups G'_2 do not form a five-parameter group.

THEOREM 33.—*There are six varieties of groups of collineations of type II in which the collineations of a group are not all characterized by a constant* a, *viz.* $G'_2(ABl')$, $G'_3(AB)$, $G'_3(ll')$, $G'_4(Al)$, $G'_4(Al')$, $G'_4(Bl)$.

C.—Groups of Type II of Second Class.

Returning now to the group $G'_1(ABl')a$ and using it as the fundamental group, it is possible to build up another series of groups of higher order; we shall find four varieties of such groups.

59. *The Groups $G'_2(AB)a$ and $G'_2(ll')a$.* The three-parameter group $G'_3(AB)$ is composed of ∞^1 two-parameter groups $G'_2(ABl)$, and hence contains ∞^2 one-parameter groups $G'_1(ABl')a$. Among these ∞^2 subgroups of $G'_3(AB)$, consider the ∞^1 groups G'_1 characterized by the same value of a. Take two collineations $T'(a^\beta, \beta)$ and $T'_1(a^{\beta_1}, \beta_1)$ from different one-parameter groups and combine them. The two transformations of the pencil through B are both of the kind t' and have an invariant ray in common; hence, their resultant is of the same kind and $\beta_2 = \beta + \beta_1$. The two transformations along AB and through A are in both instances of the kind t, and combine according to the law $k_2 = kk_1 = a^\beta a^{\beta_1} = a^{\beta_2}$. Therefore, the resultant T'_2 is of type II and is characterized by the same value of a as T' and T'_1. Hence, all collineations of type II, characterized by the same value of a and leaving invariant two points A and B and their joining line l, form a two-parameter group $G'_2(AB)a$.

In a similar manner, it may be shown that all collineations of type II, characterized by the same value of a and leaving invariant two lines and their point of intersection, form a two-parameter group $G'_2(ll')a$ dualistic to $G'_2(AB)a$.

60. *The Groups $G'_3(Al')a$ and $G'_3(Bl)a$.* The four-parameter group $G'_4(Al')$ or $G'_4(AB_nl')$, where B_n is in turn every point in the plane, is composed of ∞^3 one-parameter groups $G'_1(ABl')a$; ∞^2 of these one-parameter groups are characterized by the same value of a. Take from two of these groups two collineations $T'(a^\alpha, \alpha)$ and $T'_1(a^{\alpha_1}, \alpha_1)$, characterized by the same value of a, and combine them. The two transformations of the kind t' along l' combine to form one of the same kind according to the law $\alpha_2 = \alpha + \alpha_1$. The two transformations of the kind t through A, since they have the ray l' in common, combine to form one of the same kind according to the law $k_2 = kk_1 = a^\alpha a^{\alpha_1} = a^{\alpha_2}$. Hence their resultant T'_2 is also of type II and is also characterized by the same value of a. Therefore, all the collineations in the ∞^2 one-parameter groups in $G'_4(AB_nl')$, which are characterized by the same value of a, form a three-parameter group $G'_3(AB_nl')a$ or $G'_3(Al')a$. There is one such subgroup in $G'_4(Al')$ for each value of a.

In similar manner, it may be shown that the four-parameter group $G'_4(Bl)$ dualistic to $G'_4(Al')$ also breaks up into ∞^1 three-parameter subgroups $G'_3(Bl)\,a$, each characterized by a constant value of a.

61. *No Three-Parameter Subgroups of* $G'_4(Al)$. The group $G'_4(Al)$ does not break up into subgroups of the variety $G'_3(Al)\,a$. Let us take two collineations $T''(ABl')$ and $T''_1(AB_1l'_1)$ characterized by the same value of a and combine them. The invariant figures have in common the point A and the line l. The two transformations of the kind t along Al combine according to the law $a^a a^{a_1} = a^{a_2}$, so also do those of the pencil through A; for, in each case, one invariant element is common. But the two transformations of the kind t', one along Al' and the other along Al'_1, are so situated that we cannot assert that $\alpha_2 = \alpha + \alpha_1$. The same is true of the two transformations of the pencils through B and B_1; they are not in position to combine according to the law $\beta_2 = \beta + \beta_1$. Hence, subgroups of $G'_4(Al)$ of the variety $G'_3(Al)\,a$ do not exist.

THEOREM 34.—*There are five varieties of groups of collineations of type II in which all the collineations of a group are characterized by the same value of a. These groups are as follows:* $G'_1(ABl')\,a$, $\left\{\begin{matrix} G'_2(AB)\,a \\ G'_2(ll')\,a \end{matrix}\right\}$, $\left\{\begin{matrix} G'_3(Al')\,a \\ G'_3(Bl)\,a \end{matrix}\right\}$.

§4.—COLLINEATIONS OF TYPE III.

62. *Properties of a Collineation* T'''. A collineation of type III is obtained as the resultant of two elations S' and S'_1 whose invariant figures (Al) and (A_1l_1) are so situated that l_1 passes through A. Let the distance AA_1 be a and let the angle lAl_1 be θ; also, let the constants of S' and S'_1 be α and α_1.

In order to determine the character of the resultant of S' and S'_1, we must examine the resultants of the one-dimensional transformations along l_1 and through A. Along l_1, S' produces a one-dimensional transformation of the kind t', whose constant is $\alpha \sin\theta$ and S'_1 an identical transformation; hence, the resultant along l_1 is t'. Through A, S' produces an identical transformation and S'_1 a transformation of the kind t'_1 whose constant is $a\alpha_1$; hence, the resultant through A is t'_1. Therefore, the resultant of S' and S'_1 leaves invariant the point A and the line l_1 and produces along l_1 and through A one-dimensional transformations of the kind t' whose constants are respectively $\alpha \sin\theta$ and $a\alpha_1$. This collineation of type III will be designated by T'', its invariant figure by (Al) and its constants by β and $a\alpha$.

63. *The Group* $G_3''(Al)$. The groups $H_2'(A)$ and $H_2'(l)$, where A lies on l, each contain ∞^2 elations, and hence there are ∞^4 pairs of elations, one from each group, whose resultant in each case is a collineation of type III and whose invariant figure is the lineal element Al. It does not follow that the ∞^4 collineations T'' thus produced are all different; in fact, there are only ∞^3 different collineations T'' thus produced.

These ∞^3 collineations T'', each leaving Al invariant, form a three-parameter group, as may be shown. Let T'' and T_1'' be two collineations from the system leaving Al invariant. Along l the resultant of the two transformations of the kind t' is another of the same kind, since they have the invariant point A in common; through A the resultant of the two transformations of the kind t' is also another of the same kind, since they have the invariant ray l in common. Hence, the resultant of T'' and T_1'' is T_2'', a collineation of type III, leaving Al invariant. Therefore, all collineations of type III, leaving Al invariant, form a three-parameter group $G_3''(Al)$. Evidently, $G_3''(Al)$ contains $H_2'(A)$ and $H_2'(l)$ as subgroups.

THEOREM 35.—*The aggregate of all collineations of type III, leaving invariant the same lineal element Al, forms a three-parameter group* $G_3''(Al)$.

64. *The Group* $G_2''(Al)$. In article 43, it was pointed out that the system of ∞^3 of conics, touching l at A, is made up of ∞^1 nets N of ∞^2 conics each; all the conics of each net have contact of the second order at A. (All the conics of the net N have the same circle of curvature at A.) Such a net of conics is transformed by any collineation into a net of the same kind. Hence, a collineation T'' of the group $G_3''(Al)$ transforms a net N of the system into another net N_1 of the same system or into itself. Since $G_3''(Al)$ contains ∞^3 collineations T'', and there are only ∞^1 nets N in the system, it follows that each net N is transformed into itself by ∞^2 collineations T'' belonging to the group $G_3''(Al)$. All collineations T'' in $G_3''(Al)$, which leave invariant a net of conics N, form a two-parameter subgroup $G_2''(Al)$. Evidently, $G_3''(Al)$ contains ∞^1 such subgroups, one for each net N in the system of conics touching l at A.

THEOREM 36.—*The aggregate of all collineations of type III, leaving invariant the lineal element Al and a net of conics having contact of the second order at A, forms a two-parameter group* $G_2''(Al)$.

65. *The Group* $G_1''(Al)$. The system of ∞^3 conics touching l at A is composed of ∞^2 pencils of conics, each pencil being composed of ∞^1 conics having third order contact at A. Such a pencil of conics S is transformed by any collineation into another pencil of the same kind. Since there are ∞^2 such pencils S in the system of conics touching l at A and ∞^3 collineations T'' in the group $G_3''(Al)$, it follows that there are ∞^1 collineations T'' in $G_3''(Al)$ that transform any given pencil S into itself. These collineations form a one-parameter subgroup of $G_3''(Al)$. Evidently, $G_3''(Al)$ contains ∞^2 such subgroups, one for each pencil S in the system of conics touching l at A. Each net N of the system contains ∞^1 pencils S, and hence each two-parameter subgroup of $G_3''(Al)$ contains ∞^1 one parameter subgroups $G_1''(Al)$.

The conics of the pencil S are the path-curves of the group $G_1''(Al)$; for, if all the conics of S remain invariant under the collineation T'', the effect of T'' is to move each point of the plane along its path-curve. The parameter of the group $G_1''(Al)$ is α and the law of combination of the α's is $\alpha + \alpha_1 = \alpha_2$. Hence, the structure of the group $G_1''(Al)$ is the same as that of the group $H_1'(Al)$.

THEOREM 37.—*The aggregate of all collineations of type III, leaving invariant the lineal element Al and a pencil of conics having contact of the third order at A, forms a one-parameter group* $G_1''(Al)$, *whose path-curves are the conics of the pencil.*

These three are the only varieties of groups of type III. It may readily be verified that two collineations of this type, with their invariant figures in relative positions other than those discussed above, result in a collineation not of type III.

The following facts regarding the number of collineations of the different types should be noted here:

There are ∞^8 collineations of type I which form a group G_8.
" " ∞^7 " " II " do not form a group.
" " ∞^6 " " III " " " " "
" " ∞^5 " " IV " " " " "
" " ∞^4 " " V " " " " "

§5. GROUP STRUCTURE AND SINGULAR TRANSFORMATIONS.

A. Structure of the Collineation Groups of the Plane.

66. Having found a complete list of the Lie groups of collineations in the plane, we must now examine more closely into the structure of each group.

Many of these groups will be found to contain collineations of one type only, while others will be found to contain collineations of two or more types. In every case, there are collineations of a characteristic type which make up all or the greater part of the group, and among these are to be found in many cases a smaller number of collineations of one or more lower types. In most instances, the collineations of these secondary types form continuous subgroups of the given group; but sometimes these secondary collineations in a given group do not form a continuous subgroup, in which case they are called *Singular Transformations.*

The complete structure of some of these collineation groups has already been given while in other cases only the characteristic collineations of the group have been indicated. The entire list of these groups must be examined; in most cases the structure of a group will be given without proof, only in those cases where the structure is not at once evident, or where the group contains singular transformations, will the proofs be indicated. Perspective collineations constitute type IV, and elations type V.

67. *Structure of the Perspective Groups.* We found in article 7 eight varieties of perspective groups, viz. three of type V and five of type IV. The group $H'_1(Al)$ contains only collineations of type V, and $H_1(A, l)$, only those of type IV. The structure of the other six perspective groups is here indicated by symbolic equations, see article 19:

$$\begin{aligned}
H'_2(l) &= \infty^1 H'_1(\overline{A}\,l), \\
H'_2(A) &= \infty^1 H'_1(A\,\overline{l}), \\
H_2(lBl') &= \infty^1 H_1(\overline{A},\, l) + H'_1(Bl), \\
H_2(Al'B) &= \infty^1 H_1(A,\, \overline{l}) + H'_1(Al'), \\
H_3(l) &= \infty^2 H_1(\overline{A},\, l) + H'_2(l), \\
H_3(A) &= \infty^2 H_1(A,\, \overline{l}) + H'_2(A).
\end{aligned}$$

The correctness of each of these structural formulæ may be proved in detail; but it is necessary to carry out the proof in only one case. Take, for example, the group $H_2(lBl')$; we can show that this group must contain collineations of type V. Take from the group $H_2(lBl')$ two perspective collineations $S(A, l)$ and $S_1(A_1, l)$, for which the cross-ratio constants are k and $1/k$ respectively. Along the line A_1AB, where B is the intersection of l and l', we have two one-dimensional transformations of the kind t with reciprocal values of k but belonging to different subgroups having one invariant point B in common. Hence, their resultant along l' is of the kind t', and the value of α for this resultant is found to be

$$\alpha = (1 - k)\left(\frac{1}{A_1} - \frac{1}{A}\right).$$

Since both transformations along l are identical, their resultant is also identical. Hence, the resultant of S and S_1 is an elation S', which belongs to the group $H'_1(Bl)$. From the above value of α we see that by varying the value of k, or the position of the points A and A_1, all elations in the group $H'_1(Bl)$ are obtained; thus, the group $H_2(lBl')$ contains $H'_1(Bl)$ as a subgroup. In a similar manner the structural formulæ of the other perspective groups may be verified.

68. *Structure of Groups of Type III.* We found only three varieties of groups of type III, viz. $G''_1(Al)$, $G''_2(Al)$, $G''_3(Al)$, and these cases are easily disposed of. The group $G''_1(Al)$ contains only collineations of type III. The group $G''_3(Al)$ contains, as we saw, the groups $H'_2(A)$ and $H'_2(l)$. It may be proved without difficulty that the group $G''_2(Al)$ contains only collineations of type III. The structural formulæ of $G''_2(Al)$ and $G''_3(Al)$ are exhibited thus:

$$G''_2(Al) = \infty^1\, G''_1(Al).$$
$$G''_3(Al) = \infty^2\, G''_1(Al) + H'_2(A) + H'_2(l).$$

69. *Structure of Groups of Type II. First class.* There are six varieties of groups of this class; we have found, article 51, that the group $G'_2(ABl')$ contains the groups $H_1(B, l')$ and $H'_1(Al)$ as subgroups; and hence we must expect that subgroups of types IV and V will appear in the groups of higher orders of this class. The structural formulæ of these groups are as follows:

$$\begin{aligned}
G'_2(ABl') &= G'_2(ABl') + H_1(B, l') + H'_1(Al),\\
G'_3(AB) &= \infty^1 G'_2(AB\bar{l}') + H_2(BlA) + H'_2(l),\\
G'_3(ll') &= \infty^1 G'_2(A\bar{B}l') + H_2(lAl') + H'_2(A),\\
G'_4(Al') &= \infty^2 G'_2(A\bar{B}l') + H_3(l') + H'_2(A) + H'_2(l') + G''_3(Al'),\\
G'_4(Bl) &= \infty^2 G'_2(\bar{A}Bl') + (H_3 B) + H'_2(l) + H'_2(B) + G''_3(Bl),\\
G'_4(Al) &= \infty^2 G'_2(A\bar{B}\bar{l}') + \infty^2 H_1(\bar{B}, \bar{l}') + H'_2(A) + H'_2(l) + G''_3(Al).
\end{aligned}$$

In the case of the group $G'_3(AB)$, we may verify the structural formula as follows: The line l' through A may take ∞^1 different positions, and for each position of l' we have a two-parameter group $G'_2(ABl')$, all of which are contained in $G'_3(AB)$. The ∞^1 one-parameter groups $H_1(B, \bar{l}')$ in these groups

$G_2'(ABl')$, form the two-parameter group $H_2(BlA)$. The group $H_2(BlA)$ contains the subgroup $H_1'(Bl)$, and each group $G_2'(AB\bar{l}')$ contains the subgroup $H_1'(Al)$. By combining the collineations of these two groups, viz. $H_1'(Al)$ and $H_1'(Bl)$, we obtain all the elations of the group $H_2'(l)$, which is, therefore, contained in the group $G_3'(AB)$.

As another example, let us examine the structure of the group $G_4'(Al')$. The point B may take ∞^2 different positions in the plane, and for each position of B there is a group $G_2'(A\bar{B}l')$ which belongs to the group $G_4'(Al')$. The ∞^2 one-parameter perspective groups $H_1(\bar{B}, l')$, contained in these groups $G_2'(AB\bar{l}')$, form the group $H_3(l')$. The group $H_3(l')$ contains the subgroup $H_2'(l')$. The ∞^1 groups of elations $H_1'(A\bar{l})$, contained in the groups $G_2'(A\bar{B}l')$, form the group $H_2'(A)$, which is, therefore, contained in $G_4'(Al')$. Since $G_4'(Al')$ contains $H_2'(A)$ and $H_2'(l)$, it must also contain $G_3''(Al)$. Thus the structure of $G_4'(Al')$ is found. A similar course of reasoning leads to the structural formulæ of the other groups of this class.

70. *Second Class.* The group $G_1'(ABl')a$, where $a \neq 0$ or ∞, contains only collineations of type II. The structure of the other four groups of this class is shown as follows:

$$\begin{aligned}
G_2'(AB)\,a &= \infty^1 G_1'(A\bar{B}l')\,a + H_1'(Bl) + \text{S. T.},\\
G_2'(ll')\,a &= \infty^1 G_1'(\overline{AB}l')\,a + H_1'(Al') + \text{S. T.},\\
G_3'(Al')\,a &= \infty^2 G_1'(A\bar{B}l')\,a + H_2'(l') \quad + \text{S. T.},\\
G_3'(Bl)\,a &= \infty^2 G_1'(\bar{A}Bl)\,a + H_2'(B) \quad + \text{S. T.}
\end{aligned}$$

In this class of groups we meet, for the first time, with so-called singular transformations S. T.; these will be discussed later.

In order to show that the group of elations $H_1'(Bl)$ occurs in $G_2'(AB)\,a$, we take two collineations $T'(ABl')\,a$ and $T_1'(ABl_1')\,a$, whose constants are (β, a^β) and $(-\beta, a^{-\beta})$ respectively, and form their resultant. The resultant of the two one-dimensional transformations through B is identical, that along l is also identical, while that through A is of the kind t'. Hence, the resultant of T' and T_1' is an elation S' belonging to the group $H_1'(Bl)$. By varying the value of β and the positions of l' and l_1', all elations in $H_1'(Bl)$ may be produced, thus showing that all elations of the group $H_1'(Bl)$ are contained in $G_2'(AB)\,a$.

In like manner the structural formulæ for the remaining groups of this class may be verified.

71. *Structure of Groups of Type I. First Class.* The nine groups of type I, first class, show the following structural formulæ:

$$\begin{aligned}
G_2(ABC) &= G_2(ABC) + H_1(Al'') + H_1(Bl') + H_1(Cl),\\
G_3(AB\bar{l}') &= \infty^1 G_2(AB\bar{C}) + G_2'(ABl') + H_2(lAl') + H_2(AlB)\\
&\qquad + H_1(B,l') + H_1'(Al),\\
G_4(AB) &= \infty^2 G_2(AB\bar{C}) + 2\,\infty^1 G_2'(AB\bar{l}'') + H_3(l) + H_2'(l) + 2H_2(Al'B),\\
G_4(ll') &= \infty^2 G_2(A\bar{B}\bar{C}) + 2\,\infty^1 G_2'(A\bar{B}l') + H_3(A) + H_2'(A) + 2H_2(lAl'),\\
G_4(A,\,l'') &= \infty^2 G_2(A\bar{B}\bar{C}) + \infty^1 G_2'(A\bar{B}l'') + \infty^1 H_2(\bar{B}\bar{l}A)\\
&\qquad + H_1(A,\,l'') + \infty^1 H_1'(\bar{B}l),\\
G_5(Al) &= \infty^3 G_2(A\bar{B}\bar{C}) + \infty^2 G_2'(A\bar{B}\bar{l}') + \infty^2 G_2'(A\bar{B}l') + G_3''(Al) + H_3(l)\\
&\qquad + H_3(A) + H_2'(l) + H_2'(A),\\
G_6(l) &= \infty^4 G_2(ABC) + \infty^3 G_2'(\bar{A}\bar{B}l') + \infty^3 G_2'(ABl) + \infty^1 G_3''(\bar{A}l)\\
&\qquad + H_3(l) + \infty^2 H_2(\bar{A}l\bar{B}) + H_2'(l) + \infty^1 H_2'(A),\\
G_6(A) &= \infty^4 G_2(ABC) + \infty^3 G_2'(ABl) + \infty^3 G_1'(ABl') + \infty^1 G_3''(A\bar{l})\\
&\qquad + H_3(A) + \infty^2 H_2(lAl') + H_2'(A) + \infty^1 H_2'(l),\\
G_8 &= \infty^6 G_2(ABC) + \infty^5 G_2'(ABl') + \infty^3 G_3''(Al) + \infty^4 H_1(A,\,l'') + \infty^3 H_1'(Al).
\end{aligned}$$

The verification of these structural formulæ presents no special difficulties.

72. *Second Class.* The structural formulæ of the four following groups are exhibited thus:

$$\begin{aligned}
G_2(ABl')\,r &= \infty^1 G_1(AB\bar{C})\,r + H_1'(Al) + \text{S. T.},\\
G_3(AB)\,r &= \infty^2 G_1(AB\bar{C})\,r + H_2'(l) + \text{S. T.},\\
G_3(ll')\,r &= \infty^2 G_1(A\bar{B}\bar{C})\,r + H_2'(A) + \text{S. T.},\\
G_4(Al)\,r &= \infty^3 G_1(A\bar{B}\bar{C})\,r + H_2'(A) + H_2'(l) + G_3''(Al) + \text{S. T.}
\end{aligned}$$

From the group $G_2(ABl')\,r$ take two collineations $T(ABC)r$ and $T(ABC_1)r$, where C and C_1 are collinear with A, whose cross-ratio constants are respectively k and $1/k$. The resultants along AB and through A are identical, while those along Al' and through B are of the kind t'. The constant of the resultant along Al' is given by $\alpha = (1-k)\left(\frac{1}{AC_1} - \frac{1}{AC}\right)$. Hence, the resultant of T and T_1 is an elation and belongs to the group $H_1'(Al)$. By varying k and the positions of C and C_1, α can be made to take on all values; hence, all elations in the group $H_1'(Al)$ are contained in $G_2(ABl')\,r$.

Again, from the group $G_3(AB)r$, take two collineations $T(ABC)r$ and $T_1(ABC_1)r$, where C and C_1 are not collinear with A or B, and whose cross-ratio constants are k and $1/k$ respectively. The resultant along AB is identical, while those through A and B are both of the kind t'. Hence, the resultant of T and T_1 is $S'(Xl)$, where X is some point on l. It may be shown that all elations in $H_2'(l)$ are contained in $G_3(AB)r$.

In like manner, the formulæ of the other groups of this class may be verified.

73. *Third Class.* The list of groups of the third class shows structural formulæ as follows:

$$\begin{aligned}
G_2(AKl) &= \infty^1 G_1(ABC) + G_1''(Al),\\
G_3(K) &= \infty^2 G_1(ABC) + \infty^1 G_1''(Al),\\
G_3(S) &= \infty^2 G_1(ABC) + G_1''(Al) + H_1'(Al) + \text{S. T.},\\
G_3(A,\, l'')_{r=2} &= \infty^2 G_1(ABC) + \infty^1 H_1'(A\bar{B}\bar{l}'') + \text{S. T.},\\
G_5(l'')_{r=2} &= \infty^4 G_1(ABC) + \infty^1 H_2'(Al) + \text{S. T.},\\
G_5(A)_{r=2} &= \infty^4 G_1(ABC) + \infty^1 H_2'(l) + \text{S. T.}
\end{aligned}$$

The group $G_3(S)$ contains ∞^2 subgroups $G_1(ABC)$, where B is in turn every point on Al and C in turn every point in the plane. One conic of the system S passes through C, and BC is tangent to this conic at C. Take a collineation T from the group $G_1(ABC)$ and another T_1 from the group $G_1(AB_1C_1)$ having cross-ratio constants k and $1/k$ respectively. Their resultant is of the kind t' along Al and through A, and hence is a collineation T'' of type III belonging to the group $G_1''(Al)$. It can be shown that all the collineations of the group $G_1''(Al)$ are to be found in $G_3(S)$.

Again, take two collineations T and T_1, whose cross-ratio constants are k and $1/k$, from the groups $G_1(ABC)$ and $G_1(ABC_1)$, where C and C_1 are collinear with A. Their resultant is identical along AB and through A, but is of the kind t' along ACC_1 and through B; hence, it is an elation S' and belongs to the group $H_1'(Al)$. Evidently, all elations in $H_1'(Al)$ are contained in $G_3(S)$.

The remaining formulæ are easily verified.

B.—Singular Transformations.[*]

We come next to the consideration of singular transformations in the collineation groups of the plane; these were defined, article 66, as systems of collineations of one type not forming a continuous group, yet occurring in an otherwise continuous group of another type. We shall find two distinct kinds of singular transformations, viz. discrete systems of collineations of type V or III occurring in groups of type II, second class; and discrete systems of collineations of type II occurring in groups of type I, second and third class.

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[*] The results of the investigation of singular transformations by the present methods agree with those found by Professor Taber and his pupils, Rettger, Slocum, and Williams. See Taber, Bull. Amer. Math. Soc., Vol. VI, p. 199; Rettger, Amer. Jour. Math., Vol. XXII, p. 60; Williams, Proc. Amer. Acad., Vol. XXXV, p. 97; Slocum, Proc. Amer. Acad., Vol. XXXVI, p. 92.

74. *Singular Transformations in $G_2'(AB)a$ and $G_2'(ll')a$.* It was shown, article 70, that the group $G_2'(AB)a$ contains $H_1'(Bl)$ as a subgroup. It also contains other collineations of type V, as will be shown. Take two collineations $T'(ABl')$ and $T_1'(ABl_1')$, whose constants are (β, a^β) and (β_1, a^{β_1}); for every value of $a \neq 0$ or ∞, β and β_1 may be so chosen that $\beta + \beta_1 \neq 0$, while $a^{\beta+\beta_1} = 1$. Put $a = re^{i\theta}$ and $\beta + \beta_1 = p + iq$, then we have

$$(p + iq)(\log r + i\theta) = 2n\pi i,$$

whence $$p \log r - q\theta = 0 \text{ and } q \log r + p\theta = 2n\pi;$$

also, $$p = \frac{2n\pi\theta}{\log^2 r + \theta^2} \quad\text{and}\quad q = \frac{2n\pi \log r}{\log^2 r + \theta^2},$$

Hence, for any given value of a, β and β_1 can always be found to satisfy the conditions

$$\beta + \beta_1 \neq 0 \quad\text{and}\quad a^{\beta+\beta_1} = 1.$$

The resultant of T' and T_1' is identical along AB, but the resultant one-dimensional transformations through A and B are both of the kind t'. Hence, the resultant is an elation $S'(Xl)$, where X is some point on AB, not A or B. In the above expressions for p and q, n has only integral values; hence, the resultant elations S' do not form a continuous system. For any value of $a \neq 0$ or ∞, $\beta + \beta_1$ cannot be infinitesimal and satisfy the above conditions so that all the elations in $H_2'(l)$ cannot be obtained in this manner. Hence, $G_2'(AB)a$ contains, besides the group $H_1'(Bl)$, a system of elations $S'(Xl)$ not forming a continuous group; these are singular transformations.

In exactly the same manner it may be shown that the group $G_2'(ll')a$ contains, besides the subgroup of elations $H_1'(Al')$, a discrete system of elations $S'(A\lambda)$ not forming a continuous group. These are, therefore, singular transformations in $G_2'(ll')a$.

THEOREM 38.—*The groups $G_2'(AB)a$ and $G_2'(ll')a$ each contain a discrete system of elations which are singular transformations of the groups.*

75. *Singular Transformations in $G'_3(Bl)a$ and $G'_3(Al')a$.* The group $G'_3(Bl)a$ contains ∞^1 subgroups $G'_2(AB)a$, one for each point A on the line l and the group $H'_2(B)$; it, therefore, contains all the singular transformations of type V to be found in its subgroups $G'_2(AB)a$. Take two collineations $T'(ABl')$ and $T'_1(A_1Bl'_1)$ whose constants are (β, a^{β}) and (β_1, a^{β_1}). If β and β_1 are so chosen that $\beta + \beta_1 = 0$, the resultant of T' and T'_1 is identical through B and of the kind t' along l; hence, it is an elation belonging to the group $H'_2(B)$. But if $\beta + \beta_1 \neq 0$ and $a^{\beta + \beta_1} = 1$, the resultants through B and along l are both of the kind t'; hence, it is a collineation of type III and belongs to the group $G''_1(Bl)$. For the same reasons given above, no infinitesimal collineation of type III can be produced in this manner, and hence all collineations in $G''_1(Bl)$ are not to be found in $G'_3(Bl)a$. These singular collineations of Type III in $G'_3(Bl)a$ also appear as the resultants of the singular collineations of type V in $G'_2(Bl)a$ and the elations of the group $H'_2(B)$.

In like manner it can be shown that the group $G'_3(Al')a$ contains singular transformations of types V and III.

THEOREM 39.—*The groups $G'_3(Bl)a$ and $G'_3(Al')a$ each contain discrete systems of collineations of both types III and V which are singular transformations of the groups.*

76. *Singular Transformations in $G_2(ABl')r$.* We shall next examine the group $G_2(ABl')r$ for singular transformations. There are two cases to be considered according as r is a rational or an irrational number. When r is rational, the group $G_1(ABC)r$ contains (art. 26), one perspective involutoric collineation S, whose vertex and axis are respectively a vertex and the opposite side of the invariant triangle (ABC).

Suppose that r has odd numerator and odd denominator; then, according to the notation agreed upon, S has its vertex at B and its axis along AC. For such a value of r the group $G_2(ABl')r$ contains ∞^1 subgroups $G_1(ABC)r$; but these all contain the same perspective collineation S; hence, the group $G_2(ABl')r$ contains but one perspective involutoric collineation S. Let S be combined with any collineation of type I in $G_2(ABl')r$; the resultant one-dimensional transformations along both AB and AC are of the kind t, and hence, the resultant of S and T is of type I and is one of the collineations of the group $G_2(ABl')r$. On the other hand, let S be combined with S', a collineation of type V in $H'_1(Al)$

(which is a subgroup of $G_2(ABl')r$). Along AB we have an involutoric combined with an identical transformation, and the resultant in this direction is involutoric with invariant points at A and B. Along AC we have an identical combined with a transformation of the kind t', and the resultant is of the kind t'. Through B we have an identical combined with a transformation of the kind t', and the resultant is also of the kind t'. Hence, the resultant of S and S' is T', a collineation of type II, whose invariant figure is (ABl') and whose constants k and α are $k = -1$ and α equal the α of S. If S be combined in turn with each elation of $H_1'(Al)$, we have an infinite system of collineations of type II, for each of which $k = -1$ while α has all complex values. The parameters k and α of this system of collineations of type II do not satisfy the relation $k = a^\alpha$, and hence (article 49) do not form a continuous group. These collineations of type II in $G_2(ABl')r$ are singular transformations. The structure of $G_2(ABl')r$ is thus seen to be

$$G_2(ABl')r = \infty\, G_1(AB\overline{C})r + H_1'(Al) + \infty^1 T'.$$

Let us next examine for singular transformations the group $G_2(ABl')r$, where r is rational with odd numerator and even denominator. Each subgroup $G_1(ABC)r$ of $G_2(ABl')r$ contains one perspective involutoric collineation S; these are all different, and form a system (S) whose common axis is AB and whose vertices are in turn every point on l'. Any one of these perspective collineations, combined with a collineation of type I in $G_2(ABl')r$, results in a collineation of type I also belonging to $G_2(ABl')r$. The resultant of any collineation of the system (S) with any elation of the group $H_1'(Al)$ is another perspective collineation S_1 of the system (S). The resultant of any two perspective collineations of the system (S) is an elation belonging to the group $H_1'(Al)$. Thus the group $G_2(ABl')r$, where r is rational with odd numerator and even denominator, contains ∞^1 involutoric perspective collineations, but no singular transformations of type II. These perspective collineations are not singular transformations in the sense of the definition, for each of them belongs to a subgroup of $G_2(ABl')r$.

In like manner, it may be shown that the group $G_2(ABl')r$, where r is rational with even numerator and odd denominator, contains ∞^1 involutoric perspective collineations having a common vertex at A and axes in turn every line through B; but it contains no singular transformations. The group $G_2(ABl')r$, where r is irrational, contains (article 27) no involutoric perspective collineation and hence no singular transformations.

From these results, we conclude that the group $G_2(ABl')\,r$ contains singular transformations of type II when, and only when, its subgroups have one common involutoric perspective collineation S. This depends on the manner in which the invariant triangles of the subgroups $G_1(ABC)\,r$ are put together. Thus, when the line l' passes through A, the group $G_2(ABl')\,r$ contains singular transformations when r has odd numerator and odd denominator. On the other hand, if l' passes through B, in which case it is designated by l'', the group $G_2(ABl'')\,r$ has singular transformations when r has even numerator and odd denominator.

THEOREM 40.—*The group $G_2(ABl')\,r$, where r is a rational number such that the subgroups of $G_2(ABl')\,r$ all contain the same involutoric perspective collineation S, contains singular transformations; these are of type II, and all have the same value of k, viz. $k = -1$.*

77. *Other Groups Containing Singular Transformations of Type II.* Any collineation group of the plane which contains no subgroup of type II and which contains subgroups of the variety $G_2(ABl')\,r$ such that each contains but a single perspective involutoric collineation S, will evidently contain singular transformations of type II. In addition to $G_2(ABl')\,r$, the following groups of type I, second class, also contain singular transformations: $G_3(AB)\,r$, $G_3(ll')\,r$, $G_4(Al)\,r$.

In type I, third class, the group $G_3(S)$ contains ∞^4 subgroups $G_2(ABl')\,r = -1$. Each of these subgroups contains ∞^1 singular transformations, and hence $G_3(S)$ contains ∞^2 singular collineations of type II.

The group $G_3(A, l'')$ contains ∞^1 subgroups $G_2(ABl'')\,r = 2$. There is evidently but one involutoric perspective collineation S in the group $G_3(A, l'')$, and this has its vertex at A and its axis coinciding with l''; S belongs to every subgroup $G_2(ABl'')\,r = 2$ in $G_3(A, l'')$. S, combined in turn with each of the ∞^2 elations in $G_3(A, l'')$, gives ∞^2 singular transformations of type II.

The group $G_5(l'')$ contains ∞^2 subgroups $G_3(A, l'')$, one for each position of A in the plane; hence, it contains ∞^3 subgroups $G_2(ABl'')\,r = 2$. Since each of these two-parameter subgroups contains ∞^1 singular transformations of type II, it follows that the group $G_5(l'')$ contains ∞^4 such singular transformations.

The group $G_5(A)$, dualistic to $G_5(l'')$, also contains ∞^4 singular transformations of type II.

THEOREM 41.—*The following groups, and no others, contain singular transformations of type II:* $G_2(ABl')\,r$, $G_3(AB)\,r$, $G_3(ll')\,r$, *and* $G_4(Al)\,r$ (*when* r *is rational*); $G_3(S)$, $G_3(A, l'')_{r=2}$, $G_5(l'')_{r=2}$ *and* $G_5(A)_{r=2}$.

§6.—MIXED GROUPS OF COLLINEATIONS IN THE PLANE.

78. In this section the following problems will be investigated: To find (1) all collineations in the plane that leave fixed one vertex of a triangle and interchange the other two vertices; (2) that interchange a pair of points; (3) that interchange a pair of lines; (4) that permute the vertices of a triangle; (5) that permute the vertices of a quadrilateral; (6) that permute the vertices of a polygon of n sides.

The first four of these problems lead us to the consideration of certain mixed groups of collineations. For example, all collineations, leaving a triangle invariant as a whole, form a mixed group $mG(ABC)$, which consists of the continuous group $G_2(ABC)$ and all other collineations interchanging a pair of its vertices and also those permuting the three vertices.

79. *The Mixed Groups* $mG_1(PQ)$ *and* $mG_1(ll')$. As a necessary preliminary, we must first determine all one-dimensional projective transformations that interchange a pair of points on a line. There are ∞^1 projective transformations of the points on a line that leave invariant a pair of points P and Q; these form a continuous group $G_1(PQ)$. The aggregate of all transformations of both kinds, i. e. those that leave P and Q separately invariant and those that interchange P and Q, forms the mixed group $mG_1(PQ)$.

Every transformation interchanging P and Q is an involutoric transformation of the kind t, and its cross-ratio is $k = -1$. If the invariant points of t are A and B, they must be so situated that the cross-ratio $(ABPQ) = -1$. There are ∞^1 pairs of points AB having this relation; hence, there are ∞^1 involutoric transformations t interchanging P and Q. Since no transformation of the kind t' can be involutoric, it follows that there are no transformations of the kind t' interchanging P and Q,

It is evident that similar results hold also for the mixed group of one-dimensional transformations, leaving invariant and interchanging a pair of lines.

THEOREM 42.—*The mixed group* $\begin{Bmatrix} mG_1(PQ) \\ mG_1(ll') \end{Bmatrix}$ *contains* ∞^1 *involutoric trans-*

formations in addition to the transformations of the continuous group $\left\{\begin{matrix} G_1(PQ) \\ G_1(ll') \end{matrix}\right\}$; *it contains no other transformations.*

80. *The Mixed Group* mG_2 (APQ). Let (APQ) be any three points forming a triangle. Any collineation that interchanges P and Q must leave the line PQ invariant. Let (ABC) be the invariant triangle of a collineation of type I which interchanges P and Q; then two of the vertices, as B and C, must be on the line PQ and so situated that the cross-ratio $(BCPQ) = -1$. All collineations in the group $G_2(ABC)$, for which the cross-ratio along the side BC is -1, leave A invariant and interchange P and Q; they belong, therefore, to the mixed group $mG_2(APQ)$.

Let k and k' be the two independent parameters of the continuous group $G_2(ABC)$, and let k be taken along AB and k' along AC. Since the product of the three cross-ratios taken in the same order around the triangle must be unity, we have $(k)(-1)(1/k') = 1$; thus, $k + k' = 0$. Hence, out of the ∞^2 collineations in $G_2(ABC)$, where $(BCPQ) = -1$, there are ∞^1 that satisfy the condition $k + k' = 0$ and interchange P and Q. The pair of points BC can be chosen in ∞^1 different ways, so that $(BCPQ) = -1$; if out of each of these groups $G_2(ABC)$ we select the collineations that satisfy the relation $k + k' = 0$, we obtain ∞^2 collineations of type I that leave A invariant and interchange P and Q.

Among these ∞^2 collineations of type I interchanging P and Q, there are ∞^1 of type IV. In every group $G_2(ABC)$, there are two collineations, viz. $k = 1$, $k' = -1$ and $k = -1$, $k' = 1$ which satisfy the condition $k + k' = 0$ and are not of type I. They are involutoric perspective collineations with the vertex always on the line PQ.

Since no collineation of types III or V is involutoric along an invariant line, it follows that the mixed group $mG_2(APQ)$ contains no collineations of these types. A collineation of type II, whose invariant figure is (BCl'), may be involutoric along BC, but it cannot belong to the mixed group $mG_2(APQ)$, for the point A would then have to be a second invariant point on the line l', which is impossible.

THEOREM 43.—*The mixed group* $mG_2(APQ)$ *contains, in addition to the continuous group* $G_2(APQ)$, ∞^2 *collineations of type I and* ∞^1 *of type IV, which leave* A *invariant and interchange* P *and* Q.

81. *The Mixed Groups $mG_4(PQ)$ and $mG_4(ll')$.* The continuous group $G_4(PQ)$ contains ∞^4 collineations, leaving P and Q separately invariant; we seek, in addition to these, all collineations which interchange P and Q. There are ∞^3 triangles (ABC) so situated that $BCPQ$ are collinear and the cross-ratio $(BCPQ) = -1$. Each of these triangles is the invariant triangles of a two-parameter group in which ∞^1 collineations satisfy the relation $k + k' = 0$, and hence interchange P and Q. Therefore, there are ∞^4 collineations of type I which interchange P and Q.

Let us consider the group $G_2'(BCl')$ of type II, where $(BCPQ) = -1$, The collineations of this group depend upon two-parameters k and α. When $k = -1$, the transformations along BC are involutoric and interchange P and Q. The group $G_2'(BCl')$ contains ∞^1 such collineations, one for each value of α. The figure (BCl') can be chosen in ∞^2 different positions satisfying the condition $(BCPQ) = -1$. Hence, there are ∞^3 collineations of type II which interchange P and Q.

Let g be any line of the plane cutting PQ in B and take C on PQ such that $(BCPQ) = -1$. The involutoric collineation of the group $H_1(C, g)$ interchanges P and Q. There are evidently ∞^2 such involutoric collineations, one for each line of the plane not passing through P or Q.

In like manner, it may be shown that the mixed group $mG_4(ll')$ has a similar structure to $mG_4(PQ)$; these groups are dualistic, and the properties of the former may be inferred at once from those of the latter.

THEOREM 44.—*The mixed group* $\left\{\begin{matrix} mG_4(PQ) \\ mG_4(ll') \end{matrix}\right\}$ *contains, besides the continuous group* $\left\{\begin{matrix} G_4(PQ) \\ G_4(ll') \end{matrix}\right\}$, *$\infty^4$ collineations of type I, ∞^3 of type II and ∞^2 of type IV, which interchange* $\left\{\begin{matrix} P \\ l \end{matrix}\right\}$ *and* $\left\{\begin{matrix} Q \\ l' \end{matrix}\right\}$.

82. *Collineations which Permute the Vertices of a Triangle.* Let P, Q, R be the vertices of a triangle; we wish to find all collineations which change P into Q, Q into R and R into P; also, their inverses, viz. those that change P into R, R into Q and Q into P. Let K be any conic circumscribing the triangle; with PQR as the triangle of reference, the homogeneous equation of K may be written

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 0. \qquad (1)$$

Let x', y', z' be the coordinates of any point A; the polar of A with respect to the triangle of reference is

$$\frac{x}{x'}+\frac{y}{y'}+\frac{z}{z'}=0. \tag{2}$$

The polar of A with respect to the conic K is given by

$$x(bz'+cy')+y(az'+cx')+z(ay'+bx')=0. \tag{3}$$

We wish to determine the point x', y', z', so that its polars with respect to the triangle (PQR) and the conic K coincide. Comparing equations (2) and (3), we find $x':y':z'=a:b:c$.

When the point A is not on a side of the triangle of reference, we find one, and only one, position of A such that its polar with respect to the triangle is at the same time its polar with respect to the conic K. The converse of this proposition is also true; if we choose any point A and take its polar l with respect to the triangle of reference, we shall find one, and only one, conic K circumscribing the triangle for which A and l are pole and polar.

The line l and conic K, whose equations are respectively

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0 \quad\text{and}\quad \frac{a}{x}+\frac{b}{y}+\frac{c}{z}=0,$$

intersect in a pair of points B and C. (ABC) is the invariant triangle of a two-parameter group $G_2(ABC)$; the path-curves of its one-parameter subgroup $G_1(ABC)\,r=2$ are conics and K is one of these path-curves. Consequently, all collineations of the group $G_1(ABC)\,r=2$ transform K into itself.

The coordinates of A are $(a,\,b,\,c)$ of $B(a,\,\omega b,\,\omega^2 c)$, of $C(a,\,\omega^2 b,\,\omega c)$, where $\omega^3=1$. The lines AB, AC, AP, AQ, AR are given by the following equations respectively:

$$\frac{x}{a}+\frac{y}{\omega b}+\frac{z}{\omega^2 c}=0,\quad \frac{x}{a}+\frac{y}{\omega^2 b}+\frac{z}{\omega c}=0,$$

$$\frac{y}{b}=\frac{z}{c},\quad \frac{x}{a}=\frac{z}{c},\quad \frac{x}{a}=\frac{y}{b}, \tag{4}$$

From these equations we readily find that the cross-ratios of the three pencils $A(BCPQ)$ $A(BCQR)$, $A(BCRP)$ are all equal to each other and each equal to ω, i. e. $e^{\frac{2\pi i}{3}}$. Consequently, the collineations of the group $G_1(ABC)\,r=2$, for which $k=e^{\frac{2\pi i}{3}}$ changes P into Q, Q into R, and R into P. The inverse of this collineation, for which $k=e^{\frac{4\pi i}{3}}$, changes P into R, R into Q, and Q into P.

There are ∞^2 conics circumscribing the triangle (PQR); for each of these conics there is a point A and a line l which are pole and polar with respect to both triangle and conic. Consequently, there are ∞^2 two-parameter groups $G_2(ABC)$, each of which contains a pair of inverse collineations that permute the vertices of the triangle (PQR). Each of these collineations is of order 3.

THEOREM 45.—*There are ∞^2 collineations of type I and period 3 that permute the vertices of a triangle.*

83. *Collineations which Permute the Vertices of a Quadrangle.* Let $(PQRS)$ be four points in a plane forming the vertices of a quadrangle; let PR and QS meet in A, PQ and RS in A'', and PS and QR in A'. Let PR and QS meet $A'A''$ in U and V respectively. On the line $A'A''$ the two pairs of points $A'A''$ and UV determine an involution whose double points are designated by B and C. The two-parameter group $G_2(ABC)$ contains three collineations which permute the vertices of the quadrangle $(PQRS)$.

In order to verify this statement, we reason thus: Let T be a collineation permuting $PQRS$ in the order named; we then have the substitution T,

$$T: \left\{ \begin{matrix} P & Q & R & S \\ Q & R & S & P \end{matrix} \right\}.$$

Since the lines PQ and RS are changed into QR and SP respectively, A'', the point of intersection of the first pair, is transformed into A' the point of intersection of the second pair. In like manner, it is shown that A' is transformed into A''. Thus the line $A'A''$ is an invariant line of T, since a pair of points on it are interchanged. It may be shown, in the same way, that the lines PR and QS are interchanged; hence, their point of intersection A is an invariant point of T. Since PR and QS are interchanged while $A'A''$ remains invariant, it follows that U and V must be interchanged. Hence, the invariant triangle of T is (ABC) and the transformation is involutoric along BC.

From the above considerations, we see that T must belong to the group $G_2(ABC)$ and be one of the collineations of that group satisfying the condition $k + k' = 0$. It must also be of period 4; hence, $k^4 = 1$, i. e. $k = i, -i, -1, 1$. Taking for T that collineation for which $k = i$, its inverse is T^3 for which $k = -i$. The collineation T^2 is involutoric perspective and leaves invariant A and every point on $A'A''$.

There are two other triangles which bear to the quadrangle $(PQRS)$ the same relation as the triangle (ABC). On the line AA'', the two pairs of points

AA'' and the intersection of AA'' with PS and QR determine an involution whose double points may be designated as B' and C'. The two-parameter group $G_2(A'B'C')$ contains three collineations which also permute the vertices of the quadrangle $(PQRS)$; also for these $k=i, -i, -1, 1$. Another group $G_2(A''B''C'')$, where B'' and C'' are the double points of the involution on the line AA' determined by the pairs of points AA' and the intersection of AA' with PQ and RS, contains three collineations which also permute the vertices of the quadrangle $(PQRS)$.

THEOREM 46.—*There are (including the identical collineation) ten collineations in the plane permuting the vertices of a quadrangle* $(PQRS)$.

84. *Group of Invariant Pencil of Conics.* We are now in position to enumerate all the collineations that transform into itself a pencil of conics through four points. All collineations, leaving invariant the set of four points $(PQRS)$, also leaves invariant as a whole the pencil of conics through these points. There are two collineations of type I and period three that leave invariant one of the points as S and permute the other three; hence, there are in all eight of this character. There are also six collineations of type I and period four that permute the four points. In addition to these there are nine involutoric perspective collineations, i. e. of type IV and period two, which interchange the points in pairs, and the identical collineation. Thus there are, all told, twenty-four collineations which transform into itself a pencil of conics through four points. These form a discontinuous group of order twenty-four $G_{24}(PQRS)$. This group is abstractly identical with the well-known substitution group on four letters.

In a similar manner, it may be shown that there is a dualistic group of order twenty-four which leaves invariant a range of conics touching four lines $(pqrs)$.

THEOREM 47.—*There are twenty-four collineations of the plane which leave invariant a* $\left\{\begin{matrix}\textit{pencil}\\ \textit{range}\end{matrix}\right\}$ *of conics* $\left\{\begin{matrix}\textit{through}\\ \textit{touching}\end{matrix}\right\}$ *four* $\left\{\begin{matrix}\textit{points}\\ \textit{lines}\end{matrix}\right\}$, *these form a group of order twenty-four,* $\left\{\begin{matrix}G_{24}(PQRS)\\ G_{24}(pqrs)\end{matrix}\right\}$.

§7.—TABLE OF GROUPS OF COLLINEATIONS OF THE PLANE.

In this table, the collineation groups of the plane are classified according to the five types of collineations. Each group is designated by an appropriate symbol. The self-dualistic groups are enclosed in brackets, thus: $\{G_2(ABC)\}$; a pair of dualistic groups are bracketed together, thus: $\left\{\begin{matrix} G_6(A) \\ G_6(l) \end{matrix}\right\}$. Similar tables are given by Lie, "Continuierliche Gruppen," pp. 288–291, and by Meyer, "Chicago Congress Papers," pp. 188–190; but in these tables of Lie and Meyer the notation and classification is entirely different. The numbers on the right refer to Lie's and Meyer's tables respectively.

TYPE I.

A.—Groups of the First Class.

	Symbol.	Invariant Figure.	Lie.	Meyer.
(1).	$\{G_2(ABC)\}$.	Triangle	(31)	(23)
(2).	$\{G_3(ABl')\}$.	Two points, their joining line, and a line through one of these points	(20)	(15)
(3).	$\left\{\begin{matrix} G_4(AB) \\ G_4(ll') \end{matrix}\right\}$.	Two points and their joining line	(11)	(9a)
(4).		Two lines and their point of intersection,	(12)	(9b)
(5).	$\{G_4(A, l'')\}$.	Point and line, separate	(10)	(8)
(6).	$\{G_5(Al)\}$.	Lineal element	(6)	(4)
(7).	$\left\{\begin{matrix} G_6(A) \\ G_6(l) \end{matrix}\right\}$.	Point	(3)	(2b)
(8).		Line	(2)	(2a)
(9).	$\{G_8\}$.	No invariant figure	(1)	(1)

B.—Groups of the Second Class.

	Symbol.	Invariant Figure.	Lie.	Meyer.
(1).	$\{G_1(ABC)r\}$.	Triangle and pencil of path-curves	(35)	(26)
(2).	$\{G_2(ABl')r\}$.	Same as $G_3(ABl')$	(27)	(20)
(3).	$\left\{\begin{matrix} G_3(AB)r \\ G_3(ll')\ r \end{matrix}\right\}$.	Same as $G_4(AB)$	(18)	(14a)
(4).		Same as $G_4(ll')$	(19)	(14b)
(5).	$\{G_4(Al)r\}$.	Same as $G_5(Al)$	(7)	(5)

C.—Groups of the Third Class.

	Symbol.	Invariant Figure.	Lie.	Meyer.
(1).	$\{G_1(K_n)$.	Triangle and pencil of conics having double contact	(35)	(26)
(2).	$\{G_2(AKl)\}$.	Conic and point A on it and tangent at A,	(34)	(25)
(3).	$\{G_3(K)\}$.	Conic	(23)	(17)

		Invariant Figure.	Lie.	Meyer.
(4).	$\{G_3(S)\}$.	Lineal element Al and pencil of conics having third order contact at A and touching l	(14)	(11)
(5).	$\{G_3(A, l'')r=2\}$.	Same as $G_4(A, l'')$	(17)	(13)
(6).	$\{G_4(Al)r=-1\}$.	Lineal element	(8)	(6)
(7).	$\{G_5(A)\,r=2\}$.	Point	(5)	(3b)
(8).	$\{G_5(l)\,r=2\}$.	Line	(4)	(3a)

TYPE II.

A.—Groups of the First Class.

(1).	$\{G_2'(ABl')\}$.	Same as $G_3(ABl')$	(28)	(21)
(2).	$\{G_3'(AB)\}$.	Same as $G_4(AB)$	(18)	(14a)
(3).	$\{G_3'(ll')\}$.	Same as $G_4(ll')$	(19)	(14b)
(4).	$\{G_4'(Al')\}$.	Lineal element	(7)	(5a)
(5).	$\{G_4'(Bl)\}$.	" "	(7)	(5b)
(6).	$\{G_4'(Al)\}$.	" "	(9)	(7)

B.—Groups of the Second Class.

(1).	$\{G_1'(ABl')a\}$.	Same as $G_3(ABl')$ and pencil of path-curves	(36)	(27)
(2).	$\{G_2'(AB)a\}$.	Same as $G_4(AB)$	(25)	(19a)
(3).	$\{G_2'(ll')a\}$.	Same as $G_4(ll')$	(26)	(19b)
(4).	$\{G_3'(Al')a\}$.	Lineal element	(15)	(12a)
(5).	$\{G_3'(Bl)a\}$.	" "	(16)	(12b)

TYPE III.

(1).	$\{G_1''(Al)\}$.	Lineal element Al and pencil of ∞^1 conics having contact of third order with l at A	(37)	(28)
(2).	$\{G_2''(Al)\}$.	Lineal element Al and net of ∞^2 conics having contact of second order with l at A	(24)	(18)
(3).	$\{G_3''(Al)\}$.	Lineal element Al and ∞^3 conics touching l at A	(13)	(10)

TYPE IV.

		Invariant Figure.	Lie.	Meyer.
(1).	$\{H_1(A,\, l)\}$.	All points on l and all lines through A . .	(38)	(29)
(2).	$H_2(lBl')$.	The lines l and l' and all points on l	(32)	(24a)
(3).	$H_2(Al'B)$.	The points A and B and all lines through B	(33)	(24b)
(4).	$H_3(A)$.	All lines through A	(22)	(16b)
(5).	$H_3(l)$.	All points on l	(21)	(16a)

TYPE V.

(1).	$\{H'_1(Al)\}$.	All points on l and all lines through A . .	(39)	(30)
(2).	$H'_2(A)$.	All lines through A	(30)	(22b)
(3).	$H'_2(l)$.	All points on l	(29)	(22a)

A comparison of the tables of Lie and Meyer show them to be practically identical; in fact, Meyer has only put Lie's infinitesimal notation into finite form. A comparison of the present table with Lie's table shows some results worthy of notice. The most important result reached here is that every group is assigned to its proper type and class. This is a distinct advance upon Lie's treatment of the projective group.

The groups numbered (18) and (19) in Lie's table are the groups $G_3(AB)r$ and $G_3(ll')r$ for all values of r. These are, in general, of type I, second class, but they include also the groups $G'_3(AB)$ and $G'_3(ll')$ of type II, first class. The existence of these latter groups would hardly be suspected from Lie's or Meyer's formulæ.

Again, Lie's group (7) is, in general, $G_4(Al)r$ for all values of r, but it also includes the groups $G'_4(Al')$ and $G'_4(Bl)$. The existence of the latter as distinct groups is unknown to Lie's theory.

The group $G_4(Al)\,r = -1$, (8) of Lie's table, is only a special case of $G_4(Al)r$, (7) of the same table; and, though its structure is somewhat peculiar, it is doubtful if it is worthy of special mention in the list.

LAWRENCE, KANSAS, *May* 9, 1901.

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